Mathematics 1052, Calculus II
Exam 1, April 3rd, 2010

1. (8 points) If an unknown function y satisfies the equation

$$y' = \frac{2x}{\sqrt[3]{x^2+4}}$$

with the condition that $y(2) = -1$, then what is y?

Solution: We must integrate y' against dx to find y up to a constant. We will use substitution $u = x^2+4$ and $du = 2x\,dx$

$$y = \int \frac{2x}{\sqrt[3]{x^2+4}}dx = \int \frac{1}{\sqrt[3]{u}}du = \int u^{-1/3}\,du = \frac{3}{2}u^{2/3} + c = \frac{3(x^2+4)^{2/3}}{2} + c$$

We also know that when $x = 2$ we have $y = -1$. Then

$$-1 = \frac{3(2^2+4)^{2/3}}{2} + c = 6 + c$$

which implies $c = -7$. This means

$$y = \frac{3(x^2+4)^{2/3}}{2} - 7$$

2. Compute the following integrals

(a) (5 points) $\displaystyle\int_e^{\sqrt{e}} \frac{1}{v\ln^2 v}dv$

Solution: Use substitution $u = \ln(v)$. Note that $\ln^2(v)$ means $(\ln v)^2$ NOT $\ln(2v)$. Our substitution indicates $du = \dfrac{dv}{v}$, and therefore $dv = v\,du$. We must also change the boundary points as suggested by the substitution. So, when $v = e$ we have $u = \ln(e) = 1$ and when $v = \sqrt{e} = e^{1/2}$ we have $u = \ln(e^{1/2}) = \frac{1}{2}$. Then

$$\int_e^{\sqrt{e}} \frac{1}{v\ln^2 v}dv = \int_1^{1/2} \frac{1}{v\,u^2}v\,du = \int_1^{1/2} u^{-2} = \left.\frac{u^{-1}}{-1}\right|_1^{1/2} = -1$$

(b) (5 points) $\displaystyle\int \cos^2(4x)dx$

Solution: We will use one of the double angle formulas

$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$

This formula tells us that

$$\int \cos^2(4x)dx = \frac{1}{2}\int (1+\cos(8x))\,dx = \frac{x}{2}+\frac{1}{2}\cdot\frac{\sin(8x)}{8}+c = \frac{x}{2}+\frac{\sin(8x)}{16}+c$$

Note that the last 8 in the denominator comes from *the reverse chain rule*.

(c) (5 points) $\displaystyle\int_0^{\ln(2\sqrt{3})} \frac{e^x}{4+e^{2x}}dx$

Solution: We use the substitution $u = e^x$. Observe that $e^{2x} = (e^x)^2$, and therefore, it can be replaced by u^2. Moreover, $du = e^x dx = u\,dx$ and $dx = \frac{du}{u}$. We also change the boundary points: when $x = 0$ we have $u = e^0 = 1$ and when $x = \ln(2\sqrt{3})$ we have $u = e^{\ln(2\sqrt{3})} = 2\sqrt{3}$. Then

$$\int_0^{\ln(2\sqrt{3})} \frac{e^x}{4+e^{2x}}dx = \int_1^{2\sqrt{3}} \frac{u}{4+u^2}\frac{du}{u} = \int_1^{2\sqrt{3}} \frac{1}{2^2+u^2}du = \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Bigg|_1^{2\sqrt{3}}$$
$$=\frac{1}{2}\arctan(\sqrt{3}) - \frac{1}{2}\arctan(1/2) = \frac{\pi}{6} - \frac{1}{2}\arctan(1/2)$$

3. (10 points) Evaluate the following limit by interpreting it as a Riemann sum form of an integral, and then computing that integral:

$$\lim_{n\to\infty} \frac{1}{n}\left(\cos\left(\frac{\pi}{n}\right)+\cos\left(\frac{2\pi}{n}\right)+\cdots+\cos\left(\frac{n\pi}{n}\right)\right)$$

Solution: By looking at the Δx term, which is $\dfrac{1}{n}$, and the individual choice points $\dfrac{i\pi}{n}$ where $i = 1,2,\ldots,n$, we see that the interval is $[0,1]$. Then the function we need to integrate is $\cos(\pi x)$. Therefore the limit above represents the integral

$$\int_0^1 \cos(\pi x)\,dx = \frac{1}{\pi}\sin(\pi x)\Big|_0^1 = 0$$

4. (6 points) What is $F'(x)$ if $F(x) = \int_{\tan(x)}^{1} \sqrt{\sin(t)}dt$?

> **Solution:** We use the Fundamental Theorem of Calculus together with the Chain Rule to get
> $$\frac{d}{dx}\int_{\tan(x)}^{1} \sqrt{\sin(t)}dt = \sqrt{\sin(1)}\frac{d\,1}{dx} - \sqrt{\sin(\tan(x))}\frac{d\,tan(x)}{dx}$$
> which gives us
> $$-\sqrt{\sin(\tan(x))}\sec^2(x)$$

5. (15 points) Consider the area enclosed by the line $y = x+2$, the parabola $y = 4-x^2$ and the x-axis.

   (a) Formulate (but do not compute) the area as an integral over x.

> **Solution:** As you can see from the graph above, the area must be computed by two different integrals. For that we need the intersection of the curves $y = 4-x^2$ and $y = x+2$. We obtain that point by setting these equations equal to each other.
> $$4-x^2 = x+2 \implies 0 = x^2+x-2 \implies (x+2)(x-1) = 0$$
> We need both intersection points $x = -2$ and $x = 1$. We also need the intersection points of $y = 4-x^2$ with the x-axis:
> $$4-x^2 = 0 \implies x = \pm 2$$
> Then the area is
> $$\int_{-2}^{2}(x+2-0)\,dx + \int_{1}^{2}(4-x^2-0)\,dx$$

   (b) Formulate (but do not compute) the area as an integral over y.

> **Solution:** In order to write the integral which computes the area over y, we must view the area as a collection of *horizontal* (as opposed to *vertical*) line segments, and compute the lengths of these line segments as a function of y. For that, we need to write our graphs where x is the dependent variable and y is the independent variable. We obtain
> $$y = x+2 \implies x = y-2$$
> for the first curve, while for the other curve we get
> $$y = 4-x^2 \implies x^2 = 4-y \implies x = \pm\sqrt{4-y}$$

The solution with the $+$ sign is the right half of the parabola, and the solution with the $-$ sign is the left half of the same parabola. Notice that, we need the y-coordinate of the intersection point to write the new integral. Since $x = 1$, by using either of the curves we get $y = 3$. Then the area is

$$\int_0^3 (\sqrt{4-y} - (y-2))\, dy$$

(c) Evaluate one of the integrals above.

Solution: We will compute both integrals here for the purpose of demonstration. First the integral over x

$$\int_{-2}^{1} (x+2)\, dx + \int_1^2 (4-x^2)\, dx = \frac{x^2}{2} + 2x\Big|_{-2}^{1} + 4x - \frac{x^3}{3}\Big|_1^2 = \frac{37}{6}$$

Now, the integral over y: here we use a substitution $u = 4-y$ and $du = -dy$. Also $y = 0$ is replaced by 4, and $y = 3$ is replaced by $u = 1$ as boundary points.

$$\int_0^3 (\sqrt{4-y} - (y-2))\, dy = \int_4^1 (\sqrt{u} - ((4-u)-2))\,(-du) = \int_1^4 (u^{1/2} + u - 2)\, du$$
$$= \frac{2}{3}u^{3/2} + \frac{u^2}{2} - 2u\Big|_1^4 = \frac{37}{6}$$

6. (15 points) Evaluate the indefinite integral $\int \ln(1-x^2)dx$

Solution: We will use the method of *Integration By Parts*: we set $f = \ln(1-x^2)$ and $dg = dx$. Then

$$df = \frac{-2x}{1-x^2} \quad \text{and} \quad g = x$$

and

$$\int \ln(1-x^2)\, dx = x\ln(1-x^2) - \int \frac{-2x^2}{-x^2+1}\, dx = x\ln(1-x^2) - \int \frac{2x^2}{x^2-1}\, dx$$

Now, we will use method of *Partial Fractions*. For that, we first need to factorize the denominator as $x^2 - 1 = (x-1)(x+1)$ and reduce the degree of the polynomial in the numerator by Euclidean long division:

$$2x^2 = 2(x^2-1) + 2$$

Then the second part of our integral is equal to

$$\int \frac{2x^2}{x^2-1}\,dx = \int 2 + \frac{2}{(x-1)(x+1)}\,dx = 2x + \int \left(\frac{A}{x-1} + \frac{B}{x+1}\right)\,dx$$

In order to get the unknown coefficients A and B, we must solve

$$2 = A(x+1) + B(x-1) \implies (A+B) = 0 \text{ and } A - B = 2$$

and therefore $A = 1$ and $B = -1$. Thus the remaining part of our integral is

$$\int \frac{1}{x-1}\,dx - \frac{1}{x+1}\,dx = \ln|x-1| - \ln|x+1| + c$$

which makes our final answer

$$\int \ln(1-x^2)\,dx = x\ln(1-x^2) - 2x - \ln|x-1| + \ln|x+1| + c$$

7. Compute the following integrals

(a) (9 points) $\int \sin^7(\theta)\cos^3(\theta)d\theta$

Solution: There are two equally valid similar solutions. Here we give just one solution. We separate one of the cosine terms as use it for $du = \cos(\theta)d\theta$ and therefore we assume $u = \sin(\theta)$. Notice that we write

$$\int \sin^7(\theta)\cos^3(\theta)\,d\theta = \int \sin^7(\theta)\cos^2(\theta)\cos(\theta)d\theta$$

and make a subsititution $u = \sin(\theta)$ we will have an extra $\cos^2(\theta)$. This term must be rewritten in terms of $\sin(\theta)$. Bu we have the simple identity $\cos^2(\theta) = 1 - \sin^2(\theta)$. Then our integral transforms into

$$\int u^7(1-u^2)\,du = \int (u^7 - u^9)\,du = \frac{u^8}{8} - \frac{u^{10}}{10} + c = \frac{\sin^8(\theta)}{8} - \frac{\sin^{10}(\theta)}{10} + c$$

(b) (10 points) $\int \frac{\sqrt{x^2-9}}{x}dx$

Solution: The most obvious solution is by the method of *Trigonometric Substitution* and we use $x = 3\sec(\theta)$ because our integral contains $\sqrt{x^2-3^2}$. Then we first have

$dx = 3\sec(\theta)\tan(\theta)\,d\theta$ and

$$\sqrt{x^2-9} = \sqrt{9\sec^2(\theta)-9} = \sqrt{9\tan^2(\theta)} = 3\tan(\theta)$$

Now, our integral can be written as

$$\int \frac{\sqrt{x^2-9}}{x}dx = \int \frac{3\tan(\theta)}{3\sec(\theta)} 3\sec(\theta)\tan(\theta)\,d\theta = 3\int \tan^2(\theta)\,d\theta$$
$$=3\int (\sec^2(\theta)-1)\,d\theta = 3\tan(\theta) - 3\theta + c$$

Since $\sec(\theta) = \frac{x}{3}$, we see $\theta = \operatorname{arcsec}(x/3)$. Now, we draw right triangle with an inner angle θ which satisfies $\cos(\theta) = \frac{3}{x}$. This implies $\tan(\theta) = \frac{\sqrt{x^2-9}}{3}$. Then our final answer for this question is

$$\sqrt{x^2-9} - 3\operatorname{arcsec}(x/3) + c$$

8. (12 points) Evaluate $\displaystyle\int \frac{4x^2-3x+2}{x^2(x-1)}dx$

Solution: We will use the method of *Partial Fractions*. We first notice that there is a repeated factor in the denominator. Therefore, the fractional function $\frac{4x^2-3x+2}{x^2(x-1)}$ must split as

$$\frac{4x^2-3x+2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

In order to solve for A, B and C we must first make a common denominator. This leads us to the equation

$$4x^2-3x+2 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B$$

Then we immediately conclude that $B = -2$. Also $B - A = -3$ which means $A = 1$. And finally, $A + C = 4$ which yields $C = 3$. Now, we can compute our integral:

$$\int \frac{4x^2-3x+2}{x^2(x-1)}dx = \int \left(\frac{1}{x} - \frac{2}{x^2} + \frac{3}{x-1}\right)dx = \ln|x| + \frac{2}{x} + 3\ln|x-1| + \text{arbitrary constant}$$

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Bonus: Show that the integrals

$$I_n = \int x^n e^{-x} dx$$

satisfy the recursion formula

$$I_n = -x^n e^{-x} + n \cdot I_{n-1} \quad \text{where} \quad I_0 = -e^{-x} + c$$

for any natural number $n \geq 1$. Now, using this recursion formula, evaluate

$$\int x^3 e^{-x} dx$$

Solution: For the recursion relations, we will try to compute our integrals using the method of *Integration by Parts*. We first identify $f = x^n$ and $dg = e^{-x}dx$. Then $df = nx^{n-1}$ and $g = -e^{-x}$, and therefore

$$I_n = \int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx = -x^n e^{-x} + n I_{n-1}$$

as we wanted to show. For $n = 0$ we see that

$$I_0 = \int x^0 e^{-x} dx = \int e^{-x} dx = -e^{-x} + c$$

Now, using these formulas

$$\begin{aligned} I_3 &= -x^3 e^{-x} + 3 I_2 = -x^3 e^{-x} + 3(-x^2 e^{-x} + 2I_1) = -x^3 e^{-x} - 3x^2 e^{-x} + 6I_1 \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6(-xe^{-x} + I_0) = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c \\ &= -e^{-x}(x^3 + 3x^2 + 6x + 6) + c \end{aligned}$$