1. (8 points) If an unknown function \( y \) satisfies the equation
\[
y' = \frac{2x}{\sqrt{x^2 + 4}}
\]
with the condition that \( y(2) = -1 \), then what is \( y \)?

**Solution:** We must integrate \( y' \) against \( dx \) to find \( y \) up to a constant. We will use substitution \( u = x^2 + 4 \) and \( du = 2x \, dx \)

\[
y = \int \frac{2x}{\sqrt{x^2 + 4}} \, dx = \int \frac{1}{\sqrt{u}} \, du = \int u^{-1/2} \, du = \frac{3}{2} u^{1/2} + c = \frac{3(x^2 + 4)^{1/2}}{2} + c
\]

We also know that when \( x = 2 \) we have \( y = -1 \). Then

\[
-1 = \frac{3(2^2 + 4)^{1/2}}{2} + c = 6 + c
\]

which implies \( c = -7 \). This means

\[
y = \frac{3(x^2 + 4)^{1/2}}{2} - 7
\]

2. Compute the following integrals

(a) (5 points) \( \int_1^e \frac{1}{v \ln^2 v} \, dv \)

**Solution:** Use substitution \( u = \ln(v) \). Note that \( \ln^2(v) \) means \( \ln(v)^2 \) NOT \( \ln(2v) \). Our substitution indicates \( du = \frac{dv}{v} \), and therefore \( dv = v \, du \). We must also change the boundary points as suggested by the substitution. So, when \( v = e \) we have \( u = \ln(e) = 1 \) and when \( v = \sqrt{e} = e^{1/2} \) we have \( u = \ln(e^{1/2}) = \frac{1}{2} \). Then

\[
\int_1^e \frac{1}{v \ln^2 v} \, dv = \int_1^{1/2} \frac{1}{u^2} \, v \, du = \int_1^{1/2} u^{-2} \, du = \left. \frac{u^{-1}}{-1} \right|_1^{1/2} = -1
\]

(b) (5 points) \( \int \cos^2(4x) \, dx \)

**Solution:** We will use one of the double angle formulas

\[
\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))
\]
This formula tells us that
\[ \int \cos^2(4x) \, dx = \frac{1}{2} \int (1 + \cos(8x)) \, dx = \frac{x}{2} + \frac{1}{2} \frac{\sin(8x)}{8} + c = \frac{x}{2} + \frac{\sin(8x)}{16} + c \]
Note that the last 8 in the denominator comes from the reverse chain rule.

(c) (5 points) \[ \int_0^{\ln(2\sqrt{3})} \frac{e^x}{4 + e^{2x}} \, dx \]

**Solution:** We use the substitution \( u = e^x \). Observe that \( e^{2x} = (e^x)^2 \), and therefore, it can be replaced by \( u^2 \). Moreover, \( du = e^x \, dx = u \, dx \) and \( dx = \frac{du}{u} \). We also change the boundary points: when \( x = 0 \) we have \( u = e^0 = 1 \) and when \( x = \ln(2\sqrt{3}) \) we have \( u = e^{\ln(2\sqrt{3})} = 2\sqrt{3} \). Then
\[ \int_0^{\ln(2\sqrt{3})} \frac{e^x}{4 + e^{2x}} \, dx = \int_1^{2\sqrt{3}} \frac{u \, du}{4 + u^2} u = \int_1^{2\sqrt{3}} \frac{1}{2} \frac{1}{u^2 + 2^2} \, du = \frac{1}{2} \text{arctan} \left( \frac{u}{2} \right) \bigg|_1^{2\sqrt{3}} \]
\[ = \frac{1}{2} \text{arctan}(\sqrt{3}) - \frac{1}{2} \text{arctan}(1/2) = \frac{\pi}{6} - \frac{1}{2} \text{arctan}(1/2) \]

3. (10 points) Evaluate the following limit by interpreting it as a Riemann sum form of an integral, and then computing that integral:
\[ \lim_{n \to \infty} \frac{1}{n} \left( \cos \left( \frac{\pi}{n} \right) + \cos \left( \frac{2\pi}{n} \right) + \cdots + \cos \left( \frac{n\pi}{n} \right) \right) \]

**Solution:** By looking at the \( \Delta x \) term, which is \( \frac{1}{n} \), and the individual choice points \( \frac{i\pi}{n} \) where \( i = 1, 2, \ldots, n \), we see that the interval is \( [0, 1] \). Then the function we need to integrate is \( \cos(\pi x) \). Therefore the limit above represents the integral
\[ \int_0^1 \cos(\pi x) \, dx = \frac{1}{\pi} \sin(\pi x) \bigg|_0^1 = 0 \]
4. (6 points) What is $F'(x)$ if $F(x) = \int_{\tan(x)}^{1} \sqrt{\sin(t)} dt$?

**Solution:** We use the Fundamental Theorem of Calculus together with the Chain Rule to get

$$\frac{d}{dx} \int_{\tan(x)}^{1} \sqrt{\sin(t)} dt = \sqrt{\sin(1)} \frac{d}{dx} \tan(x) - \sqrt{\sin(\tan(x))} \frac{d}{dx} \tan(x)$$

which gives us

$$-\sqrt{\sin(\tan(x))} \sec^2(x)$$

5. (15 points) Consider the area enclosed by the line $y = x + 2$, the parabola $y = 4 - x^2$ and the $x$-axis.

(a) Formulate (but do not compute) the area as an integral over $x$.

**Solution:** As you can see from the graph above, the area must be computed by two different integrals. For that we need the intersection of the curves $y = 4 - x^2$ and $y = x + 2$. We obtain that point by setting these equations equal to each other.

$$4 - x^2 = x + 2 \implies 0 = x^2 + x - 2 \implies (x + 2)(x - 1) = 0$$

We need both intersection points $x = -2$ and $x = 1$. We also need the intersection points of $y = 4 - x^2$ with the $x$-axis:

$$4 - x^2 = 0 \implies x = \pm 2$$

Then the area is

$$\int_{-2}^{1} (x + 2 - 0) dx + \int_{1}^{2} (4 - x^2 - 0) dx$$

(b) Formulate (but do not compute) the area as an integral over $y$.

**Solution:** In order to write the integral which computes the area over $y$, we must view the area as a collection of horizontal (as opposed to vertical) line segments, and compute the lengths of these line segments as a function of $y$. For that, we need to write our graphs where $x$ is the dependent variable and $y$ is the independent variable. We obtain

$$y = x + 2 \implies x = y - 2$$

for the first curve, while for the other curve we get

$$y = 4 - x^2 \implies x^2 = 4 - y \implies x = \pm \sqrt{4 - y}$$
The solution with the + sign is the right half of the parabola, and the solution with the − sign is the left half of the same parabola. Notice that, we need the y-coordinate of the intersection point to write the new integral. Since \( x = 1 \), by using either of the curves we get \( y = 3 \). Then the area is

\[
\int_0^3 (\sqrt{4-y} - (y - 2)) \, dy
\]

(c) Evaluate one of the integrals above.

**Solution:** We will compute both integrals here for the purpose of demonstration. First the integral over \( x \)

\[
\int_{-2}^1 (x+2) \, dx + \int_1^2 (4-x^2) \, dx = \frac{x^2}{2} + 2x \bigg|_{-2}^1 + 4x - \frac{x^3}{3} \bigg|_1^2 = \frac{37}{6}
\]

Now, the integral over \( y \): here we use a substitution \( u = 4 - y \) and \( du = -dy \). Also \( y = 0 \) is replaced by 4, and \( y = 3 \) is replaced by \( u = 1 \) as boundary points.

\[
\int_0^3 (\sqrt{4-y} - (y - 2)) \, dy = \int_4^1 (\sqrt{u} - ((4 - u) - 2)) \, (-du) = \int_1^4 (u^{1/2} + u - 2) \, du
\]

\[
= \frac{2}{3} u^{3/2} + \frac{u^2}{2} - 2u \bigg|_{1}^{4} = \frac{37}{6}
\]

6. (15 points) Evaluate the indefinite integral \( \int \ln(1-x^2) \, dx \)

**Solution:** We will use the method of Integration By Parts: we set \( f = \ln(1-x^2) \) and \( dg = dx \). Then

\[
df = -\frac{2x}{1-x^2} \quad \text{and} \quad g = x
\]

and

\[
\int \ln(1-x^2) \, dx = x \ln(1-x^2) - \int -\frac{2x^2}{x^2+1} \, dx = x \ln(1-x^2) - \int \frac{2x^2}{x^2-1} \, dx
\]

Now, we will use method of Partial Fractions. For that, we first need to factorize the denominator as \( x^2 - 1 = (x-1)(x+1) \) and reduce the degree of the polynomial in the numerator by Euclidean long division:

\[
2x^2 = 2(x^2 - 1) + 2
\]
Then the second part of our integral is equal to

\[ \int \frac{2x^2}{x^2-1} \, dx = \int 2 + \frac{2}{(x-1)(x+1)} \, dx = 2x + \int \left( \frac{A}{x-1} + \frac{B}{x+1} \right) \, dx \]

In order to get the unknown coefficients \( A \) and \( B \), we must solve

\[ 2 = A(x + 1) + B(x - 1) \implies (A + B) = 0 \quad \text{and} \quad A - B = 2 \]

and therefore \( A = 1 \) and \( B = -1 \). Thus the remaining part of our integral is

\[ \int \frac{1}{x-1} \, dx - \frac{1}{x+1} \, dx = \ln |x - 1| - \ln |x + 1| + c \]

which makes our final answer

\[ \int \ln(1 - x^2) \, dx = x \ln(1 - x^2) - 2x - \ln |x - 1| + \ln |x + 1| + c \]

7. Compute the following integrals

(a) (9 points) \( \int \sin^7(\theta) \cos^3(\theta) \, d\theta \)

**Solution:** There are two equally valid similar solutions. Here we give just one solution. We separate one of the cosine terms as use it for \( du = \cos(\theta) \, d\theta \) and therefore we assume \( u = \sin(\theta) \). Notice that we write

\[ \int \sin^7(\theta) \cos^3(\theta) \, d\theta = \int \sin^7(\theta) \cos^2(\theta) \, d\theta \]

and make a substitute \( u = \sin(\theta) \) we will have an extra \( \cos^2(\theta) \). This term must be rewritten in terms of \( \sin(\theta) \). But we have the simple identity \( \cos^2(\theta) = 1 - \sin^2(\theta) \). Then our integral transforms into

\[ \int u^7(1-u^2) \, du = \int (u^7 - u^9) \, du = \frac{u^{10}}{10} - \frac{u^8}{8} + c = \frac{\sin^8(\theta)}{8} - \frac{\sin^{10}(\theta)}{10} + c \]

(b) (10 points) \( \int \frac{\sqrt{x^2 - 9}}{x} \, dx \)

**Solution:** The most obvious solution is by the method of *Trigonometric Substitution* and we use \( x = 3 \sec(\theta) \) because our integral contains \( \sqrt{x^2 - 3^2} \). Then we first have
dx = 3 sec(θ) tan(θ) dθ and

\[ \sqrt{x^2 - 9} = \sqrt{9 sec^2(\theta) - 9} = \sqrt{9 tan^2(\theta)} = 3 tan(\theta) \]

Now, our integral can be written as

\[ \int \frac{\sqrt{x^2 - 9}}{x} dx = \int \frac{3 tan(\theta)}{3 sec(\theta)} \cdot 3 sec(\theta) tan(\theta) d\theta = 3 \int tan^2(\theta) d\theta \]

\[ = 3 \int (sec^2(\theta) - 1) d\theta = 3 tan(\theta) - 3\theta + c \]

Since sec(θ) = \( \frac{\sqrt{x^2 - 9}}{x} \), we see θ = arcsec(x/3). Now, we draw right triangle with an inner angle θ which satisfies cos(θ) = \( \frac{3}{x} \). This implies tan(θ) = \( \sqrt{x^2 - 9} \). Then our final answer for this question is

\[ \sqrt{x^2 - 9} - 3 \text{arcsec}(x/3) + c \]

8. (12 points) Evaluate \( \int \frac{4x^2 - 3x + 2}{x^2(x-1)} dx \)

**Solution:** We will use the method of Partial Fractions. We first notice that there is a repeated factor in the denominator. Therefore, the fractional function \( \frac{4x^2 - 3x + 2}{x^2(x-1)} \) must split as

\[ \frac{4x^2 - 3x + 2}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \]

In order to solve for A, B and C we must first make a common denominator. This leads us to the equation

\[ 4x^2 - 3x + 2 = Ax(x-1) + B(x-1) + Cx^2 = (A+C)x^2 + (B-A)x - B \]

Then we immediately conclude that B = -2. Also B - A = -3 which means A = 1. And finally, A + C = 4 which yields C = 3. Now, we can compute our integral:

\[ \int \frac{4x^2 - 3x + 2}{x^2(x-1)} dx = \int \left( \frac{1}{x} - \frac{2}{x^2} + \frac{3}{x-1} \right) dx = \ln |x| + \frac{2}{x} + 3 \ln |x-1| + \text{arbitrary constant} \]
**Bonus:** Show that the integrals

\[ I_n = \int x^n e^{-x} \, dx \]

satisfy the recursion formula

\[ I_n = -x^n e^{-x} + n \cdot I_{n-1} \quad \text{where} \quad I_0 = -e^{-x} + c \]

for any natural number \( n \geq 1 \). Now, using this recursion formula, evaluate

\[ \int x^3 e^{-x} \, dx \]

**Solution:** For the recursion relations, we will try to compute our integrals using the method of *Integration by Parts*. We first identify \( f = x^n \) and \( dg = e^{-x} \, dx \). Then \( df = n x^{n-1} \) and \( g = -e^{-x} \), and therefore

\[ I_n = \int x^n e^{-x} \, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} \, dx = -x^n e^{-x} + n I_{n-1} \]

as we wanted to show. For \( n = 0 \) we see that

\[ I_0 = \int x^0 e^{-x} \, dx = \int e^{-x} \, dx = -e^{-x} + c \]

Now, using these formulas

\[
\begin{align*}
I_3 &= -x^3 e^{-x} + 3 I_2 = -x^3 e^{-x} + 3(-x^2 e^{-x} + 2 I_1) = -x^3 e^{-x} - 3x^2 e^{-x} + 6 I_1 \\
&= -x^3 e^{-x} - 3x^2 e^{-x} + 6(-xe^{-x} + I_0) = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c \\
&= -e^{-x}(x^3 + 3x^2 + 6x + 6) + c
\end{align*}
\]