# Mathematics 1052, Calculus II
# Exam 1, April 9th, 2011

**Name:** ______________________________
**Student Number** ______________________________

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total | Bonus |
|---|---|---|---|---|---|---|---|---|
| 10pts | 10pts | 15pts | 15pts | 15pts | 15pts | 20pts | 100pts | 15pts |
| | | | | | | | | |

This exam has 7 questions and a bonus question, for a total of 100 + 15 points on 8 pages.

*Please write your name and Bahçeşehir student number above. Read the problems carefully. You* ***must*** *show your work to get credit. Your solutions must be supported by calculations and/or explanations: no points will be given for answers that are not accompanied by supporting work. You must observe the honor code. Read and sign the following statement:*

**I have not given to, nor received any help from any of my classmates during this exam**

Signature

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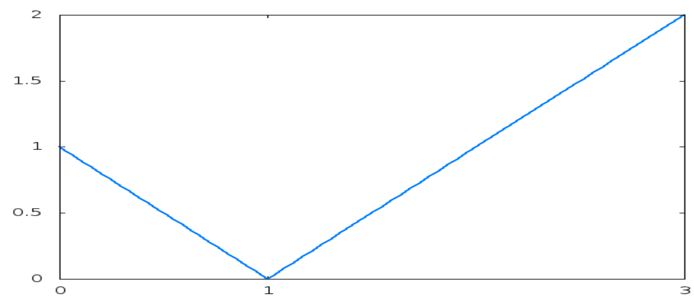
1. (10 points) Consider the function $f(x) = |x-1|$ on the interval $[0,3]$.




(a) Divide the interval into 4 pieces and compute the *lower* Riemann sum.

**Solution:** The partition points are $(0, \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3)$. For the lower Riemann Sum, we must find the $x$ values in each subinterval giving us the minimum of the function $|x-1|$. Those points are $(\frac{3}{4}, 1, \frac{3}{2}, \frac{9}{4})$. Thus the lower Rieamann Sum is

$$\frac{3}{4}(f(3/4)+f(1)+f(3/2)+f(9/4)) = \frac{3}{16}+0+\frac{3}{8}+\frac{15}{16} = \frac{3}{2}$$

(b) Divide the interval into 4 pieces and compute the *upper* Riemann sum.

**Solution:** Again, the partition points are $(0, \frac{3}{4}, \frac{3}{2}, \frac{9}{4}, 3)$. For the upper Riemann Sum, we must find the $x$ values in each subinterval giving us the maximum of the function $|x-1|$. Those points are $(0, \frac{3}{2}, \frac{9}{4}, 3)$. Thus the upper Rieamann Sum is

$$\frac{3}{4}(f(0)+f(3/2)+f(9/4)+f(3)) = \frac{3}{4}+\frac{3}{8}+\frac{15}{16}+\frac{3}{2} = \frac{57}{16}$$

2. (10 points) We have an unknown function which satisfies

$$\frac{dy}{dx} = xe^{-x^2} \text{ and } y(0) = \frac{3}{2}$$

Find $y$.

**Solution:** Since the derivative of $y$ is $xe^{-x^2}$, the unknown function $y$ is the integral of $xe^{-x^2}$. We will use ordinary substitution using $u = -x^2$. Then $du = -2xdx$.

$$y = \int xe^{-x^2} dx = \int xe^u \frac{du}{-2x} = -\frac{1}{2} \int e^u du = -\frac{e^u}{2} + c = -\frac{e^{-x^2}}{2} + c$$

Since $y(0) = \frac{3}{2}$ we see

$$y(0) = -\frac{e^0}{2} + c = -\frac{1}{2} + c = \frac{3}{2}$$

Hence $c = 2$ and $y = -\frac{e^{-x^2}}{2} + 2$.

3. (15 points) Use Leibniz Rule to find the value of $x$ that maximizes the value of the integral.

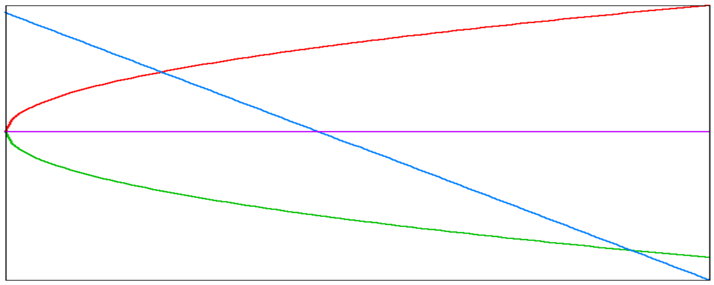
$$\int_x^{x+3} t(t-5)dt$$

**Solution:** We must find the critical points of this function which is where the derivative is 0. The derivative of this function is obtained by the Leibniz Rule.

$$\frac{d}{dx} \int_x^{x+3} t(t-5)dt = (x+3)(x-2)(1) - x(x-5)(1) = 6x - 6 = 0$$

which gives us $x = 1$. The second derivative is 6 and is positive. This tells us the critical point is a minimum. The question said “maximum” but the critical point tells us minimum. If you found the critical point, you get full points.

4. (15 points) Consider the area in the first quadrant bounded by $x = y^2$, the x-axis and $y = -x+2$.



(a) Write (but do not compute!) the area as an integral over $x$.

**Solution:**

$$\int_0^1 \sqrt{x}dx + \int_1^2 (-x+2)dx$$

(b) Write (but do not compute!) the area as an integral over $y$.

**Solution:**

$$\int_0^1 (-y+2-y^2)dy$$

(c) Compute the area using one of the integrals above.

**Solution:** The first integral gives us

$$\frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 + -\frac{1}{2}x^2+2x\Big|_1^2 = \frac{2}{3}+2-\frac{3}{2} = \frac{7}{6}$$

The second integral on the other hand yields

$$-\frac{1}{2}y^2+2y-\frac{1}{3}y^3\Big|_0^1 = -\frac{1}{2}+2-\frac{1}{3} = \frac{7}{6}$$

5. (15 points) Compute the indefinite integral $\int \frac{u^3+u^2}{u^2-9} du$

**Solution:** Do the Euclidean long division and we get

$$u^3 + u^2 = (u^2-9)(u+1) + 9u + 9$$

Then

$$\begin{aligned}\int \frac{u^3+u^2}{u^2-9} du &= \int u + 1 + \frac{9u+9}{(u-3)(u+3)} du \\ &= \int u + 1 + \frac{A}{u-3} + \frac{B}{u+3} du \\ &= \frac{u^2}{2} + u + A\ln|u-3| + B\ln|u+3| + c\end{aligned}$$

Now, let us determine the constants $A$ and $B$.

$$\begin{aligned}\frac{9u+9}{u^2-9} &= \frac{A}{u-3} + \frac{B}{u+3} \\ &= \frac{A(u+3)+B(u-3)}{u^2-9}\end{aligned}$$

This yields us

$$9u + 9 = A(u+3) + B(u-3)$$

$A = 6$ and $B = 3$.

6. (15 points) Compute the indefinite integral $\int \tan^3\theta \sec^3\theta d\theta$

**Solution:** Set $u = \sec(\theta)$ and we get $du = \sec(\theta)\tan(\theta)d\theta$.

$$\int \tan^3\theta\sec^3\theta d\theta = \int \tan^2\theta\sec^2\theta\sec\theta\tan\theta d\theta = \int (\sec^2\theta - 1)\sec^2\theta\sec\theta\tan\theta d\theta$$

After the substitution we get

$$\int (u^2-1)u^2 du = \int (u^4-u^2)du$$

which gives us

$$\frac{1}{5}u^5 - \frac{1}{3}u^3 + c = \frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta + c$$

**Solution:** (ALTERNATIVE)

$$\begin{aligned}\int \tan^3\theta\sec^3\theta d\theta &= \int \frac{\sin^3\theta}{\cos^6\theta}d\theta = \int \frac{\sin^2\theta}{\cos^6\theta}\sin\theta d\theta \\ &= \int \frac{1-\cos^2\theta}{\cos^6\theta}\sin\theta d\theta = \int \left(\frac{1}{\cos^6\theta} - \frac{1}{\cos^4\theta}\right)\sin\theta d\theta\end{aligned}$$

Now, we use the substitution $u = \cos\theta$ and $du = -\sin\theta d\theta$.

$$\begin{aligned}&= \int \left(\frac{1}{u^6} - \frac{1}{u^4}\right)(-du) = \int (u^{-4} - u^{-6})du = -\frac{1}{3}u^{-3} + \frac{1}{5}u^{-5} + c \\ &= -\frac{1}{3}\sec^3\theta + \frac{1}{5}\sec^5\theta + c\end{aligned}$$

7. (20 points) Evaluate the definite integral $\int_{1}^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}dx$

**Solution:** Use the trigonometric substitution $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$. This implies

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2|\cos\theta|$$

We also see that

$$\text{When } x = 1 \text{ we have } \sin\theta = \frac{1}{2} \text{ which implies } \theta = \frac{\pi}{6}$$

and

$$\text{When } x = \sqrt{2} \text{ we have } \sin\theta = \frac{\sqrt{2}}{2} \text{ which implies } \theta = \frac{\pi}{4}$$

Therefore, our integral reduces

$$=\frac{1}{4}\int_{\pi/6}^{\pi/4} \frac{1}{\sin^2\theta \, 2\cos\theta} 2\cos\theta d\theta = \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2\theta d\theta = -\frac{1}{4}\cot\theta\Big|_{\pi/6}^{\pi/4} = \frac{\sqrt{3}-1}{4}$$

BONUS: (15pts) Compute the following limit

$$\lim_{n\to\infty}\left(\frac{1}{1+9n}+\frac{1}{3+9n}+\frac{1}{6+9n}+\cdots+\frac{1}{3n+9n}\right)$$

**Solution:** Except the first term, the sum gives us a right Riemann Sum.

$$\frac{1}{1+9n}+\frac{1}{3+9n}+\frac{1}{6+9n}+\cdots+\frac{1}{3n+9n}=\frac{1}{1+9n}+\sum_{i=1}^{n}\frac{1}{3i+9n}$$

which we can simplify as

$$\frac{1}{1+9n}+\frac{1}{n}\ \sum_{i=1}^{n}\frac{1}{\frac{3i}{n}+9}$$

As $n$ goes to $\infty$, the first term goes to 0 and the second term converges to the integral

$$\int_0^1\frac{1}{3x+9}dx=\frac{1}{3}\ln|3x+9|\Big|_0^1=\frac{1}{3}\ln(12)-\frac{1}{3}\ln(9)$$

On the other hand we could simplify the summation in different ways. Here is another possibility

$$\frac{1}{1+9n}+\frac{1}{3+9n}+\frac{1}{6+9n}+\cdots+\frac{1}{3n+9n}=\frac{1}{1+9n}+\frac{1}{9n}\sum_{i=1}^{n}\frac{1}{\frac{i}{3n}+1}$$

where the second summand converges to

$$\frac{1}{3}\int_0^{\frac{1}{3}}\frac{dx}{1+x}$$