Mathematics 1052, Calculus II
Exam 1, April 14th, 2012

Name: ______________________________

Student Number ______________________________

1	2	3	4	5	6	7	Total
8pts	12pts	15pts	20pts	15pts	15pts	15pts	100pts

This exam has 7 questions for a total of 100 points on 6 pages.

Please write your name and Bahçeşehir student number above. Read the problems carefully. You ***must*** *show your work to get credit. Your solutions must be supported by calculations and/or explanations: no points will be given for answers that are not accompanied by supporting work. You must observe the honor code. Read and sign the following statement:*

I have not given to, nor received any help from any of my classmates during this exam

Signature

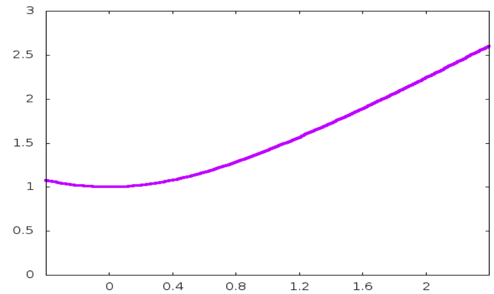
1. (8 points) Calculate the derivative

$$\frac{d}{dx}\int_{\sqrt{x^3}}^{x^2} \tan(t)\ln^3\sin(t)dt$$

Solution: We are going to use the Leibniz Rule to calculate the derivative.

$$\frac{d}{dx}\int_{\sqrt{x^3}}^{x^2} \tan(t)\ln^3\sin(t)dt = \tan(x^2)\ln^3(\sin(x^2))2x - \tan(x^{3/2})\ln^3(\sin(x^{3/2}))\frac{3}{2}\sqrt{x}$$

2. (a) (7 points) Write the lower (in this case left) Riemann Sum for the function $f(x) = \sqrt{1+x^2}$ on the interval $[0,2]$ with $n = 5$ subintervals of equal length.



Solution: Since $\sqrt{1+x^2}$ is an increasing function on the interval $[0,2]$, the lower Riemann sum is the same as left Riemann sum. When we split the interval $[0,2]$ into 5 equal pieces we get our $\Delta x = 2/5 = 0.4$, and our partition points become $(0, 0.4, 0.8, 1.2, 1.6, 2)$. Then the left Riemann sum gives us

$$0.4(\sqrt{1+0} + \sqrt{1+0.16} + \sqrt{1+0.64} + \sqrt{1+1.44} + \sqrt{1+2.56})$$

(b) (5 points) Calculate the difference between the upper and lower Riemann Sums for the set-up given in part (a). [For the same function, on the same interval, with $n = 5$.]

Solution: The upper Riemann sum is

$$0.4(\sqrt{1+0.16} + \sqrt{1+0.64} + \sqrt{1+1.44} + \sqrt{1+2.56} + \sqrt{1+4})$$

and therefore the difference is

$$0.4(\sqrt{5} - 1)$$

3. Consider the region bounded by the y-axis and the curves $y = \sin(x)$ and $y = \cos(x)$ on the interval $[0, \pi/4]$.

   (a) (5 points) Express (but do not compute) the area of the region as an integral over the x-axis.

   Solution:

   $$\int_0^{\pi/4} \cos(x) - \sin(x) \quad dx$$

   (b) (5 points) Express (but do not compute) the are of the region as an integral over the y-axis.

   Solution:

   $$\int_0^{\sqrt{2}/2} \arcsin(y)dy + \int_{\sqrt{2}/2}^{1} \arccos(y)dy$$

   (c) (5 points) Evaluate one of the integrals above.

   Solution:

   $$\int_0^{\pi/4} \cos(x) - \sin(x) \quad dx = \sin(x) + \cos(x)\Big|_0^{\pi/4}$$
   $$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1$$

4. Calculate the following indefinite integrals:

   (a) (5 points) $\displaystyle\int \frac{2x}{\sqrt{x^2+1}}dx$

   Solution: We will use a substitution: $u = x^2+1$, and we get $dx = 2x\,dx$. Then

   $$\int \frac{2x}{\sqrt{x^2+1}}dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = 2u^{1/2} + c = 2\sqrt{x^2+1} + c$$

   (b) (15 points) $\displaystyle\int \frac{2x^3}{\sqrt{x^2+1}}dx$

   Solution: Use the same substitution we have used above: $u = x^2+1$ and $du = 2xdx$. Then $x^2 = (u-1)$ and we get

   $$\begin{aligned}\int \frac{2x^3}{\sqrt{x^2+1}}dx &= \int \frac{x^2}{\sqrt{x^2+1}}2xdx = \int \frac{(u-1)}{\sqrt{u}}du \\ &= \int \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du = \int (u^{1/2} - u^{-1/2})du \\ &= \frac{2}{3}u^{3/2} - 2u^{1/2} + c = \frac{2(x^2+1)^{3/2}}{3} - 2\sqrt{x^2+1} + c\end{aligned}$$

   Solution: Use trigonometric substitution: $x = \tan(\theta)$ then $dx = \sec^2(\theta)d\theta$ and we also have $x^2+1 = \tan^2(\theta)+1 = \sec^2(\theta)$. This gives us

   $$\begin{aligned}\int \frac{2x^3}{\sqrt{x^2+1}}dx &= \int \frac{2\tan^3(\theta)}{\sec(\theta)}\sec^2(\theta)d\theta = \int 2\tan^3(\theta)\sec(\theta)d\theta \\ &= \int 2\tan^2(\theta)\tan(\theta)\sec(\theta)d\theta \\ &= \int 2(\sec^2(\theta)-1)\tan(\theta)\sec(\theta)d\theta\end{aligned}$$

   Now, perform an ordinary subsitution $u = \sec(\theta)$ and $du = \tan(\theta)\sec(\theta)d\theta$ and we get

   $$= \int 2(u^2-1)du = \frac{2u^3}{3} - 2u + c = \frac{2\sec^3(\theta)}{3} - 2\sec(\theta) + c$$

   Now, sketch a right triangle with $\tan(\theta) = x$ and read $\sec(\theta)$ as $\sqrt{x^2+1}$. Then we get the result as

   $$\frac{2(x^2+1)^{3/2}}{3} - 2\sqrt{x^2+1} + c$$

5. (15 points) Calculate the indefinite integral $\int \sec^3(\theta)\tan^3(\theta)d\theta$

Solution: Use substitution $u = \sec(\theta)$ and $du = \sec(\theta)\tan(\theta)d\theta$. Now, we re-write the integral as

$$\begin{aligned}\int \sec^3(\theta)\tan^3(\theta)d\theta &= \int \sec^2(\theta)\tan^2(\theta)\sec(\theta)\tan(\theta)d\theta \\ &= \int \sec^2(\theta)(\sec^2(\theta)-1)\sec(\theta)\tan(\theta)d\theta \\ &= \int u^2(u^2-1)du = \int (u^4-u^2)\ du = \frac{u^5}{5} - \frac{u^3}{3} + c \\ &= \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + c\end{aligned}$$

6. (15 points) Calculate the definite integral $\int_0^1 \arcsin(x)dx$.

Solution: We will use integration by parts with $f = \arcsin(x)$ and $dg = dx$. Then we get $df = \dfrac{dx}{\sqrt{1-x^2}}$ and $g = x$. This yields

$$\int_0^1 \arcsin(x)dx = x\arcsin(x)\Big|_0^1 - \int_0^1 \frac{xdx}{\sqrt{1-x^2}} = \frac{\pi}{2} - \int_0^1 \frac{xdx}{\sqrt{1-x^2}}$$

Now, we use a substitution $u = 1-x^2$ and $du = -2xdx$. Along with it, we change the boundaries. At $x = 0$ we have $u = 1$ and at $x = 1$ we have $u = 0$. Then

$$= \frac{\pi}{2} + \frac{1}{2}\int_1^0 \frac{du}{\sqrt{u}} = \frac{\pi}{2} + \sqrt{u}\Big|_1^0 = \frac{\pi}{2} - 1$$

7. (15 points) Calculate the indefinite integral

$$\int \frac{x+2}{(x-2)(x^2+4)}dx$$

Solution: We will use the method of partial fractions:

$$\frac{x+2}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

and we get

$$x+2 = A(x^2+4) + (Bx+C)(x-2)$$

At $x = 2$ we get $A = \frac{1}{2}$. At $x = 0$ we get

$$2 = 4A - 2C = 2 - 2C$$

which means $C = 0$. Now, at $x = 1$ we see

$$3 = 5A - (B+C) = \frac{5}{2} - B$$

and $B = -\frac{1}{2}$. Then we get an integral of the form

$$\int \frac{x+2}{(x-2)(x^2+4)}dx = \int \frac{1}{2}\cdot\frac{1}{x-2} - \frac{1}{2}\cdot\frac{x}{x^2+4}\ dx$$

Now, for the second integral we use a substitution $u = x^2+4$ and $du = 2xdx$

$$=\frac{1}{2}\ln(x-2) - \frac{1}{4}\int\frac{du}{u}$$

$$=\frac{\ln(x-2)}{2} - \frac{\ln(u)}{4} + c = \frac{\ln(x-2)}{2} - \frac{\ln(x^2+4)}{4} + c$$