This exam has 7 questions for a total of 100 points on 6 pages.

Please write your name and Bahçeşehir student number above. Read the problems carefully. You must show your work to get credit. Your solutions must be supported by calculations and/or explanations: no points will be given for answers that are not accompanied by supporting work. You must observe the honor code. Read and sign the following statement:

I have not given to, nor received any help from any of my classmates during this exam

Signature

1. (8 points) Calculate the derivative

$$\frac{d}{dx} \int_{\sqrt{x^3}}^{x^2} \tan(t) \ln^3 \sin(t) dt$$

**Solution:** We are going to use the Leibniz Rule to calculate the derivative.

$$\frac{d}{dx} \int_{\sqrt{x^3}}^{x^2} \tan(t) \ln^3 \sin(t) dt = \tan(x^2) \ln^3(\sin(x^2))2x - \tan(x^{3/2}) \ln^3(\sin(x^{3/2})) \frac{3}{2} \sqrt{x}$$
2. (a) (7 points) Write the lower (in this case left) Riemann Sum for the function \( f(x) = \sqrt{1 + x^2} \) on the interval \([0, 2]\) with \( n = 5 \) subintervals of equal length.

Solution: Since \( \sqrt{1 + x^2} \) is an increasing function on the interval \([0, 2]\), the lower Riemann sum is the same as left Riemann sum. When we split the interval \([0, 2]\) into 5 equal pieces we get our \( \Delta x = 2/5 = 0.4 \), and our partition points become \((0, 0.4, 0.8, 1.2, 1.6, 2)\). Then the left Riemann sum gives us

\[
0.4(\sqrt{1 + 0} + \sqrt{1 + 0.16} + \sqrt{1 + 0.64} + \sqrt{1 + 1.44} + \sqrt{1 + 2.56})
\]

(b) (5 points) Calculate the difference between the upper and lower Riemann Sums for the set-up given in part (a). [For the same function, on the same interval, with \( n = 5 \).

Solution: The upper Riemann sum is

\[
0.4(\sqrt{1 + 0.16} + \sqrt{1 + 0.64} + \sqrt{1 + 1.44} + \sqrt{1 + 2.56} + \sqrt{1 + 4})
\]

and therefore the difference is

\[
0.4(\sqrt{5} - 1)
\]
3. Consider the region bounded by the $y$-axis and the curves $y = \sin(x)$ and $y = \cos(x)$ on the interval $[0, \pi/4]$.

(a) (5 points) Express (but do not compute) the area of the region as an integral over the $x$-axis.

Solution:
\[
\int_0^{\pi/4} \cos(x) - \sin(x) \, dx
\]

(b) (5 points) Express (but do not compute) the area of the region as an integral over the $y$-axis.

Solution:
\[
\int_0^{\sqrt{2}/2} \arcsin(y) \, dy + \int_{\sqrt{2}/2}^1 \arccos(y) \, dy
\]

(c) (5 points) Evaluate one of the integrals above.

Solution:
\[
\int_0^{\pi/4} \cos(x) - \sin(x) \, dx = \left[ \sin(x) + \cos(x) \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 = \sqrt{2} - 1
\]
4. Calculate the following indefinite integrals:

(a) (5 points) \( \int \frac{2x}{\sqrt{x^2 + 1}} \, dx \)

Solution: We will use a substitution: \( u = x^2 + 1 \), and we get \( dx = 2x \, du \). Then

\[
\int \frac{2x}{\sqrt{x^2 + 1}} \, dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = 2u^{1/2} + c = 2\sqrt{x^2 + 1} + c
\]

(b) (15 points) \( \int \frac{2x^3}{\sqrt{x^2 + 1}} \, dx \)

Solution: Use the same substitution we have used above: \( u = x^2 + 1 \) and \( du = 2x \, dx \). Then \( x^2 = (u - 1) \) and we get

\[
\int \frac{2x^3}{\sqrt{x^2 + 1}} \, dx = \int \frac{x^2}{\sqrt{x^2 + 1}} \, 2xdx = \int \frac{(u - 1)}{\sqrt{u}} \, du
\]

\[
= \int \left( \frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}} \right) \, du = \int (u^{1/2} - u^{-1/2}) \, du
\]

\[
= \frac{2}{3} u^{3/2} - 2u^{1/2} + c = \frac{2(x^2 + 1)^{3/2}}{3} - 2\sqrt{x^2 + 1} + c
\]

Solution: Use trigonometric substitution: \( x = \tan(\theta) \) then \( dx = \sec^2(\theta) \, d\theta \) and we also have \( x^2 + 1 = \tan^2(\theta) + 1 = \sec^2(\theta) \). This gives us

\[
\int \frac{2x^3}{\sqrt{x^2 + 1}} \, dx = \int \frac{2\tan^3(\theta)}{\sec(\theta)} \sec^2(\theta) \, d\theta = \int 2\tan^3(\theta) \sec(\theta) \, d\theta
\]

\[
= \int 2 \tan^2(\theta) \tan(\theta) \sec(\theta) \, d\theta
\]

\[
= \int 2(\sec^2(\theta) - 1) \tan(\theta) \sec(\theta) \, d\theta
\]

Now, perform an ordinary substitution \( u = \sec(\theta) \) and \( du = \tan(\theta) \sec(\theta) \, d\theta \) and we get

\[
= \int 2(u^2 - 1) \, du = \frac{2u^3}{3} - 2u + c = \frac{2\sec^3(\theta)}{3} - 2\sec(\theta) + c
\]

Now, sketch a right triangle with \( \tan(\theta) = x \) and read \( \sec(\theta) \) as \( \sqrt{x^2 + 1} \). Then we get the result as

\[
\frac{2(x^2 + 1)^{3/2}}{3} - 2\sqrt{x^2 + 1} + c
\]
5. (15 points) Calculate the indefinite integral \( \int \sec^3(\theta) \tan^3(\theta) d\theta \)

**Solution:** Use substitution \( u = \sec(\theta) \) and \( du = \sec(\theta) \tan(\theta) d\theta \). Now, we re-write the integral as

\[
\int \sec^3(\theta) \tan^3(\theta) d\theta = \int \sec^2(\theta) \tan^2(\theta) \sec(\theta) \tan(\theta) d\theta
\]

\[
= \int \sec^2(\theta) (\sec^2(\theta) - 1) \sec(\theta) \tan(\theta) d\theta
\]

\[
= \int u^2 (u^2 - 1) du = \int (u^4 - u^2) du = \frac{u^5}{5} - \frac{u^3}{3} + c
\]

\[
= \frac{\sec^5(\theta)}{5} - \frac{\sec^3(\theta)}{3} + c
\]

6. (15 points) Calculate the definite integral \( \int_0^1 \arcsin(x) dx \).

**Solution:** We will use integration by parts with \( f = \arcsin(x) \) and \( dg = dx \). Then we get \( df = \frac{dx}{\sqrt{1-x^2}} \) and \( g = x \). This yields

\[
\int_0^1 \arcsin(x) dx = x \arcsin(x) \bigg|_0^1 - \frac{x}{\sqrt{1-x^2}} \bigg|_0^1 = \frac{\pi}{2} - \frac{\pi}{2}
\]

Now, we use a substitution \( u = 1 - x^2 \) and \( du = -2xdx \). Along with it, we change the boundaries. At \( x = 0 \) we have \( u = 1 \) and at \( x = 1 \) we have \( u = 0 \). Then

\[
= \frac{\pi}{2} + \frac{1}{2} \int_1^0 \frac{du}{\sqrt{u}} = \frac{\pi}{2} + \sqrt{u} \bigg|_1^0 = \frac{\pi}{2} - 1
\]
7. (15 points) Calculate the indefinite integral
\[ \int \frac{x + 2}{(x - 2)(x^2 + 4)} \, dx \]

**Solution:** We will use the method of partial fractions:

\[ \frac{x + 2}{(x - 2)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4} \]

and we get

\[ x + 2 = A(x^2 + 4) + (Bx + C)(x - 2) \]

At \( x = 2 \) we get \( A = \frac{1}{2} \). At \( x = 0 \) we get

\[ 2 = 4A - 2C = 2 - 2C \]

which means \( C = 0 \). Now, at \( x = 1 \) we see

\[ 3 = 5A - (B + C) = \frac{5}{2} - B \]

and \( B = -\frac{1}{2} \). Then we get an integral of the form

\[ \int \frac{x + 2}{(x - 2)(x^2 + 4)} \, dx = \int \left( \frac{1}{2} \cdot \frac{1}{x - 2} - \frac{1}{2} \cdot \frac{x}{x^2 + 4} \right) \, dx \]

Now, for the second integral we use a substitution \( u = x^2 + 4 \) and \( du = 2xdx \)

\[ = \frac{1}{2} \ln(x - 2) - \frac{1}{4} \int \frac{du}{u} \]

\[ = \frac{\ln(x - 2)}{2} - \frac{\ln(u)}{4} + c = \frac{\ln(x - 2)}{2} - \frac{\ln(x^2 + 4)}{4} + c \]