# LINEAR ALGEBRA

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# Chapter 3

## DETERMINANTS

### 3.1. Introduction

In the last chapter, we have related the question of the invertibility of a square matrix to a question of solutions of systems of linear equations. In some sense, this is unsatisfactory, since it is not simple to find an answer to either of these questions without a lot of work. In this chapter, we shall relate these two questions to the question of the determinant of the matrix in question. As we shall see later, the task is reduced to checking whether this determinant is zero or non-zero. So what is the determinant?

Let us start with $1 \times 1$ matrices, of the form

$$A = (\, a \,).$$

Note here that $I_1 = (\, 1 \,)$. If $a \neq 0$, then clearly the matrix $A$ is invertible, with inverse matrix

$$A^{-1} = (\, a^{-1} \,).$$

On the other hand, if $a = 0$, then clearly no matrix $B$ can satisfy $AB = BA = I_1$, so that the matrix $A$ is not invertible. We therefore conclude that the value $a$ is a good "determinant" to determine whether the $1 \times 1$ matrix $A$ is invertible, since the matrix $A$ is invertible if and only if $a \neq 0$.

Let us then agree on the following definition.

Definition. Suppose that

$$A = (\, a \,)$$

is a $1 \times 1$ matrix. We write

$$\det(A) = a,$$

and call this the determinant of the matrix $A$.

Next, let us turn to $2 \times 2$ matrices, of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We shall use elementary row operations to find out when the matrix $A$ is invertible. So we consider the array

$$(A|I_2) = \begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}, \tag{1}$$

and try to use elementary row operations to reduce the left hand half of the array to $I_2$. Suppose first of all that $a = c = 0$. Then the array becomes

$$\begin{pmatrix} 0 & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix},$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix $I_2$. Consider next the case $a \neq 0$. Multiplying row 2 of the array (1) by $a$, we obtain

$$\begin{pmatrix} a & b & 1 & 0 \\ ac & ad & 0 & a \end{pmatrix}.$$

Adding $-c$ times row 1 to row 2, we obtain

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{pmatrix}. \tag{2}$$

If $D = ad - bc = 0$, then this becomes

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & 0 & -c & a \end{pmatrix},$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix $I_2$. On the other hand, if $D = ad - bc \neq 0$, then the array (2) can be reduced by elementary row operations to

$$\begin{pmatrix} 1 & 0 & d/D & -b/D \\ 0 & 1 & -c/D & a/D \end{pmatrix},$$

so that

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Consider finally the case $c \neq 0$. Interchanging rows 1 and 2 of the array (1), we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ a & b & 1 & 0 \end{pmatrix}.$$

Multiplying row 2 of the array by $c$, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ ac & bc & c & 0 \end{pmatrix}.$$

Adding $-a$ times row 1 to row 2, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & bc - ad & c & -a \end{pmatrix}.$$

Multiplying row 2 by $-1$, we obtain

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & ad-bc & -c & a \end{pmatrix}. \tag{3}$$

Again, if $D = ad - bc = 0$, then this becomes

$$\begin{pmatrix} c & d & 0 & 1 \\ 0 & 0 & -c & a \end{pmatrix},$$

and so it is impossible to reduce the left hand half of the array by elementary row operations to the matrix $I_2$. On the other hand, if $D = ad - bc \neq 0$, then the array (3) can be reduced by elementary row operations to

$$\begin{pmatrix} 1 & 0 & d/D & -b/D \\ 0 & 1 & -c/D & a/D \end{pmatrix},$$

so that

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Finally, note that $a = c = 0$ is a special case of $ad-bc = 0$. We therefore conclude that the value $ad-bc$ is a good "determinant" to determine whether the $2 \times 2$ matrix $A$ is invertible, since the matrix $A$ is invertible if and only if $ad - bc \neq 0$.

Let us then agree on the following definition.

DEFINITION. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a $2 \times 2$ matrix. We write

$$\det(A) = ad - bc,$$

and call this the determinant of the matrix $A$.

## 3.2. Determinants for Square Matrices of Higher Order

If we attempt to repeat the argument for $2 \times 2$ matrices to $3 \times 3$ matrices, then it is very likely that we shall end up in a mess with possibly no firm conclusion. Try the argument on $4 \times 4$ matrices if you must. Those who have their feet firmly on the ground will try a different approach.

Our approach is inductive in nature. In other words, we shall define the determinant of $2\times2$ matrices in terms of determinants of $1\times1$ matrices, define the determinant of $3\times3$ matrices in terms of determinants of $2 \times 2$ matrices, define the determinant of $4 \times 4$ matrices in terms of determinants of $3 \times 3$ matrices, and so on.

Suppose now that we have defined the determinant of $(n-1) \times (n-1)$ matrices. Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \tag{4}$$

be an $n \times n$ matrix. For every $i, j = 1, \ldots, n$, let us delete row $i$ and column $j$ of $A$ to obtain the $(n-1) \times (n-1)$ matrix

$$A_{ij} = \begin{pmatrix} a_{11} & \ldots & a_{1(j-1)} & \bullet & a_{1(j+1)} & \ldots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \ldots & a_{(i-1)(j-1)} & \bullet & a_{(i-1)(j+1)} & \ldots & a_{(i-1)n} \\ \bullet & \ldots & \bullet & \bullet & \bullet & \ldots & \bullet \\ a_{(i+1)1} & \ldots & a_{(i+1)(j-1)} & \bullet & a_{(i+1)(j+1)} & \ldots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{n(j-1)} & \bullet & a_{n(j+1)} & \ldots & a_{nn} \end{pmatrix}. \tag{5}$$

Here $\bullet$ denotes that the entry has been deleted.

Definition. The number $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the cofactor of the entry $a_{ij}$ of $A$. In other words, the cofactor of the entry $a_{ij}$ is obtained from $A$ by first deleting the row and the column containing the entry $a_{ij}$, then calculating the determinant of the resulting $(n-1) \times (n-1)$ matrix, and finally multiplying by a sign $(-1)^{i+j}$.

Note that the entries of $A$ in row $i$ are given by

$$( a_{i1} \quad \ldots \quad a_{in} ).$$

Definition. By the cofactor expansion of $A$ by row $i$, we mean the expression

$$\sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + \ldots + a_{in} C_{in}. \tag{6}$$

Note that the entries of $A$ in column $j$ are given by

$$\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

Definition. By the cofactor expansion of $A$ by column $j$, we mean the expression

$$\sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{1j} + \ldots + a_{nj} C_{nj}. \tag{7}$$

We shall state without proof the following important result. The interested reader is referred to Section 3.8 for further discussion.

**PROPOSITION 3A.** *Suppose that $A$ is an $n \times n$ matrix given by* (4). *Then the expressions* (6) *and* (7) *are all equal and independent of the row or column chosen.*

Definition. Suppose that $A$ is an $n \times n$ matrix given by (4). We call the common value in (6) and (7) the determinant of the matrix $A$, denoted by $\det(A)$.

Let us check whether this agrees with our earlier definition of the determinant of a $2 \times 2$ matrix. Writing

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we have

$$C_{11} = a_{22}, \qquad C_{12} = -a_{21}, \qquad C_{21} = -a_{12}, \qquad C_{22} = a_{11}.$$

It follows that

$$\begin{aligned} \text{by row 1 :} \quad & a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}, \\ \text{by row 2 :} \quad & a_{21}C_{21} + a_{22}C_{22} = -a_{21}a_{12} + a_{22}a_{11}, \\ \text{by column 1 :} \quad & a_{11}C_{11} + a_{21}C_{21} = a_{11}a_{22} - a_{21}a_{12}, \\ \text{by column 2 :} \quad & a_{12}C_{12} + a_{22}C_{22} = -a_{12}a_{21} + a_{22}a_{11}. \end{aligned}$$

The four values are clearly equal, and of the form $ad - bc$ as before.

EXAMPLE 3.2.1. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix}.$$

Let us use cofactor expansion by row 1. Then

$$\begin{aligned} C_{11} &= (-1)^{1+1} \det \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix} = (-1)^2(20-2) = 18, \\ C_{12} &= (-1)^{1+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = (-1)^3(5-4) = -1, \\ C_{13} &= (-1)^{1+3} \det \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} = (-1)^4(1-8) = -7, \end{aligned}$$

so that

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 36 - 3 - 35 = -2.$$

Alternatively, let us use cofactor expansion by column 2. Then

$$\begin{aligned} C_{12} &= (-1)^{1+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = (-1)^3(5-4) = -1, \\ C_{22} &= (-1)^{2+2} \det \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix} = (-1)^4(10-10) = 0, \\ C_{32} &= (-1)^{3+2} \det \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix} = (-1)^5(4-5) = 1, \end{aligned}$$

so that

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = -3 + 0 + 1 = -2.$$

When using cofactor expansion, we should choose a row or column with as few non-zero entries as possible in order to minimize the calculations.

EXAMPLE 3.2.2. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 & 5 \\ 1 & 4 & 0 & 2 \\ 5 & 4 & 8 & 5 \\ 2 & 1 & 0 & 5 \end{pmatrix}.$$

Here it is convenient to use cofactor expansion by column 3, since then

$$\det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} = 8C_{33} = 8(-1)^{3+3} \det \begin{pmatrix} 2 & 3 & 5 \\ 1 & 4 & 2 \\ 2 & 1 & 5 \end{pmatrix} = -16,$$

in view of Example 3.2.1.

## 3.3. Some Simple Observations

In this section, we shall describe two simple observations which follow immediately from the definition of the determinant by cofactor expansion.

**PROPOSITION 3B.** *Suppose that a square matrix $A$ has a zero row or has a zero column. Then* $\det(A) = 0$.

PROOF. We simply use cofactor expansion by the zero row or zero column. ◯

DEFINITION. Consider an $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

If $a_{ij} = 0$ whenever $i > j$, then $A$ is called an upper triangular matrix. If $a_{ij} = 0$ whenever $i < j$, then $A$ is called a lower triangular matrix. We also say that $A$ is a triangular matrix if it is upper triangular or lower triangular.

EXAMPLE 3.3.1. The matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

is upper triangular.

EXAMPLE 3.3.2. A diagonal matrix is both upper triangular and lower triangular.

**PROPOSITION 3C.** *Suppose that the $n \times n$ matrix*

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

*is triangular. Then $\det(A) = a_{11}a_{22} \dots a_{nn}$, the product of the diagonal entries.*

Proof. Let us assume that $A$ is upper triangular – for the case when $A$ is lower triangular, change the term "left-most column" to the term "top row" in the proof. Using cofactor expansion by the left-most column at each step, we see that

$$\det(A) = a_{11} \det \begin{pmatrix} a_{22} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n2} & \dots & a_{nn} \end{pmatrix} = a_{11}a_{22} \det \begin{pmatrix} a_{33} & \dots & a_{3n} \\ \vdots & & \vdots \\ a_{n3} & \dots & a_{nn} \end{pmatrix} = \ldots = a_{11}a_{22}\ldots a_{nn}$$

as required. ○

### 3.4. Elementary Row Operations

We now study the effect of elementary row operations on determinants. Recall that the elementary row operations that we consider are: (1) interchanging two rows; (2) adding a multiple of one row to another row; and (3) multiplying one row by a non-zero constant.

**PROPOSITION 3D.** (ELEMENTARY ROW OPERATIONS) *Suppose that $A$ is an $n \times n$ matrix.*

*(a) Suppose that the matrix $B$ is obtained from the matrix $A$ by interchanging two rows of $A$. Then* $\det(B) = -\det(A)$.

*(b) Suppose that the matrix $B$ is obtained from the matrix $A$ by adding a multiple of one row of $A$ to another row. Then* $\det(B) = \det(A)$.

*(c) Suppose that the matrix $B$ is obtained from the matrix $A$ by multiplying one row of $A$ by a non-zero constant $c$. Then* $\det(B) = c\det(A)$.

Sketch of Proof. (a) The proof is by induction on $n$. It is easily checked that the result holds when $n = 2$. When $n > 2$, we use cofactor expansion by a third row, say row $i$. Then

$$\det(B) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det(B_{ij}).$$

Note that the $(n-1) \times (n-1)$ matrices $B_{ij}$ are obtained from the matrices $A_{ij}$ by interchanging two rows of $A_{ij}$, so that $\det(B_{ij}) = -\det(A_{ij})$. It follows that

$$\det(B) = -\sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det(A_{ij}) = -\det(A)$$

as required.

(b) Again, the proof is by induction on $n$. It is easily checked that the result holds when $n = 2$. When $n > 2$, we use cofactor expansion by a third row, say row $i$. Then

$$\det(B) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det(B_{ij}).$$

Note that the $(n-1) \times (n-1)$ matrices $B_{ij}$ are obtained from the matrices $A_{ij}$ by adding a multiple of one row of $A_{ij}$ to another row, so that $\det(B_{ij}) = \det(A_{ij})$. It follows that

$$\det(B) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det(A_{ij}) = \det(A)$$

as required.

(c) This is simpler. Suppose that the matrix $B$ is obtained from the matrix $A$ by multiplying row $i$ of $A$ by a non-zero constant $c$. Then

$$\det(B) = \sum_{j=1}^{n} ca_{ij}(-1)^{i+j}\det(B_{ij}).$$

Note now that $B_{ij} = A_{ij}$, since row $i$ has been removed respectively from $B$ and $A$. It follows that

$$\det(B) = \sum_{j=1}^{n} ca_{ij}(-1)^{i+j}\det(A_{ij}) = c\det(A)$$

as required. ◯

In fact, the above operations can also be carried out on the columns of $A$. More precisely, we have the following result.

**PROPOSITION 3E.** (ELEMENTARY COLUMN OPERATIONS) *Suppose that $A$ is an $n\times n$ matrix.*

*(a) Suppose that the matrix $B$ is obtained from the matrix $A$ by interchanging two columns of $A$. Then* $\det(B) = -\det(A)$.

*(b) Suppose that the matrix $B$ is obtained from the matrix $A$ by adding a multiple of one column of $A$ to another column. Then* $\det(B) = \det(A)$.

*(c) Suppose that the matrix $B$ is obtained from the matrix $A$ by multiplying one column of $A$ by a non-zero constant $c$. Then* $\det(B) = c\det(A)$.

Elementary row and column operations can be combined with cofactor expansion to calculate the determinant of a given matrix. We shall illustrate this point by the following examples.

Example 3.4.1. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 2 & 5 \\ 1 & 4 & 1 & 2 \\ 5 & 4 & 4 & 5 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Adding $-1$ times column 3 to column 1, we have

$$\det(A) = \det\begin{pmatrix} 0 & 3 & 2 & 5 \\ 0 & 4 & 1 & 2 \\ 1 & 4 & 4 & 5 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Adding $-1/2$ times row 4 to row 3, we have

$$\det(A) = \det\begin{pmatrix} 0 & 3 & 2 & 5 \\ 0 & 4 & 1 & 2 \\ 0 & 3 & 4 & 3 \\ 2 & 2 & 0 & 4 \end{pmatrix}.$$

Using cofactor expansion by column 1, we have

$$\det(A) = 2(-1)^{4+1}\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix} = -2\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix}.$$

Adding $-1$ times row 1 to row 3, we have

$$\det(A) = -2\det\begin{pmatrix} 3 & 2 & 5 \\ 4 & 1 & 2 \\ 0 & 2 & -2 \end{pmatrix}.$$

Adding 1 times column 2 to column 3, we have

$$\det(A) = -2\det\begin{pmatrix} 3 & 2 & 7 \\ 4 & 1 & 3 \\ 0 & 2 & 0 \end{pmatrix}.$$

Using cofactor expansion by row 3, we have

$$\det(A) = -2\cdot 2(-1)^{3+2}\det\begin{pmatrix} 3 & 7 \\ 4 & 3 \end{pmatrix} = 4\det\begin{pmatrix} 3 & 7 \\ 4 & 3 \end{pmatrix}.$$

Using the formula for the determinant of $2\times 2$ matrices, we conclude that $\det(A) = 4(9-28) = -76$. Let us start again and try a different way. Dividing row 4 by 2, we have

$$\det(A) = 2\det\begin{pmatrix} 2 & 3 & 2 & 5 \\ 1 & 4 & 1 & 2 \\ 5 & 4 & 4 & 5 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

Adding $-1$ times row 4 to row 2, we have

$$\det(A) = 2\det\begin{pmatrix} 2 & 3 & 2 & 5 \\ 0 & 3 & 1 & 0 \\ 5 & 4 & 4 & 5 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

Adding $-3$ times column 3 to column 2, we have

$$\det(A) = 2\det\begin{pmatrix} 2 & -3 & 2 & 5 \\ 0 & 0 & 1 & 0 \\ 5 & -8 & 4 & 5 \\ 1 & 1 & 0 & 2 \end{pmatrix}.$$

Using cofactor expansion by row 2, we have

$$\det(A) = 2\cdot 1(-1)^{2+3}\det\begin{pmatrix} 2 & -3 & 5 \\ 5 & -8 & 5 \\ 1 & 1 & 2 \end{pmatrix} = -2\det\begin{pmatrix} 2 & -3 & 5 \\ 5 & -8 & 5 \\ 1 & 1 & 2 \end{pmatrix}.$$

Adding $-2$ times row 3 to row 1, we have

$$\det(A) = -2\det\begin{pmatrix} 0 & -5 & 1 \\ 5 & -8 & 5 \\ 1 & 1 & 2 \end{pmatrix}.$$

Adding $-5$ times row 3 to row 2, we have

$$\det(A) = -2\det\begin{pmatrix} 0 & -5 & 1 \\ 0 & -13 & -5 \\ 1 & 1 & 2 \end{pmatrix}.$$

Using cofactor expansion by column 1, we have

$$\det(A) = -2\cdot 1(-1)^{3+1}\det\begin{pmatrix} -5 & 1 \\ -13 & -5 \end{pmatrix} = -2\det\begin{pmatrix} -5 & 1 \\ -13 & -5 \end{pmatrix}.$$

Using the formula for the determinant of $2\times 2$ matrices, we conclude that $\det(A) = -2(25+13) = -76$.

Example 3.4.2. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 3 \\ 2 & 3 & 1 & 2 & 5 \\ 4 & 7 & 2 & 3 & 7 \\ 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

Here we have the least number of non-zero entries in column 3, so let us work to get more zeros into this column. Adding $-1$ times row 4 to row 2, we have

$$\det(A) = \det \begin{pmatrix} 2 & 1 & 0 & 1 & 3 \\ 1 & 3 & 0 & 1 & 2 \\ 4 & 7 & 2 & 3 & 7 \\ 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

Adding $-2$ times row 4 to row 3, we have

$$\det(A) = \det \begin{pmatrix} 2 & 1 & 0 & 1 & 3 \\ 1 & 3 & 0 & 1 & 2 \\ 2 & 7 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

Using cofactor expansion by column 3, we have

$$\det(A) = 1(-1)^{4+3} \det \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & 7 & 1 & 1 \\ 2 & 1 & 2 & 0 \end{pmatrix} = -\det \begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 3 & 1 & 2 \\ 2 & 7 & 1 & 1 \\ 2 & 1 & 2 & 0 \end{pmatrix}.$$

Adding $-1$ times column 3 to column 1, we have

$$\det(A) = -\det \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 2 \\ 1 & 7 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

Adding $-1$ times row 1 to row 3, we have

$$\det(A) = -\det \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 6 & 0 & -2 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

Using cofactor expansion by column 1, we have

$$\det(A) = -1(-1)^{1+1} \det \begin{pmatrix} 3 & 1 & 2 \\ 6 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} = -\det \begin{pmatrix} 3 & 1 & 2 \\ 6 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Adding 1 times row 1 to row 2, we have

$$\det(A) = -\det \begin{pmatrix} 3 & 1 & 2 \\ 9 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}.$$

Using cofactor expansion by column 3, we have

$$\det(A) = -2(-1)^{1+3} \det\begin{pmatrix} 9 & 1 \\ 1 & 2 \end{pmatrix} = -2 \det\begin{pmatrix} 9 & 1 \\ 1 & 2 \end{pmatrix}.$$

Using the formula for the determinant of $2 \times 2$ matrices, we conclude that $\det(A) = -2(18-1) = -34$.

EXAMPLE 3.4.3. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 4 & 1 & 0 \\ 2 & 4 & 5 & 7 & 6 & 2 \\ 4 & 6 & 1 & 9 & 2 & 1 \\ 3 & 5 & 0 & 1 & 2 & 5 \\ 2 & 4 & 5 & 3 & 6 & 2 \\ 1 & 0 & 2 & 5 & 1 & 0 \end{pmatrix}.$$

Here note that rows 1 and 6 are almost identical. Adding $-1$ times row 1 to row 6, we have

$$\det(A) = \det\begin{pmatrix} 1 & 0 & 2 & 4 & 1 & 0 \\ 2 & 4 & 5 & 7 & 6 & 2 \\ 4 & 6 & 1 & 9 & 2 & 1 \\ 3 & 5 & 0 & 1 & 2 & 5 \\ 2 & 4 & 5 & 3 & 6 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Adding $-1$ times row 5 to row 2, we have

$$\det(A) = \det\begin{pmatrix} 1 & 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 4 & 6 & 1 & 9 & 2 & 1 \\ 3 & 5 & 0 & 1 & 2 & 5 \\ 2 & 4 & 5 & 3 & 6 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Adding $-4$ times row 6 to row 2, we have

$$\det(A) = \det\begin{pmatrix} 1 & 0 & 2 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 6 & 1 & 9 & 2 & 1 \\ 3 & 5 & 0 & 1 & 2 & 5 \\ 2 & 4 & 5 & 3 & 6 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

It follows from Proposition 3B that $\det(A) = 0$.

## 3.5. Further Properties of Determinants

DEFINITION. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}.$$

By the transpose $A^t$ of $A$, we mean the matrix

$$A^t = \begin{pmatrix} a_{11} & \ldots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix}$$

obtained from $A$ by transposing rows and columns.

EXAMPLE 3.5.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

Then

$$A^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Recall that determinants of $2 \times 2$ matrices depend on determinants of $1 \times 1$ matrices; in turn, determinants of $3 \times 3$ matrices depend on determinants of $2 \times 2$ matrices, and so on. It follows that determinants of $n \times n$ matrices ultimately depend on determinants of $1 \times 1$ matrices. Note now that transposing a $1 \times 1$ matrix does not affect its determinant (why?). The result below follows in view of Proposition 3A.

**PROPOSITION 3F.** *For every $n \times n$ matrix $A$, we have* $\det(A^t) = \det(A)$.

EXAMPLE 3.5.2. We have

$$\det \begin{pmatrix} 2 & 2 & 4 & 1 & 2 \\ 1 & 3 & 7 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 2 \\ 3 & 5 & 7 & 3 & 0 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 0 & 1 & 3 \\ 2 & 3 & 1 & 2 & 5 \\ 4 & 7 & 2 & 3 & 7 \\ 1 & 0 & 1 & 1 & 3 \\ 2 & 1 & 0 & 2 & 0 \end{pmatrix} = -34.$$

Next, we shall study the determinant of a product. In Section 3.8, we shall sketch a proof of the following important result.

**PROPOSITION 3G.** *For every $n \times n$ matrices $A$ and $B$, we have* $\det(AB) = \det(A)\det(B)$.

**PROPOSITION 3H.** *Suppose that the $n \times n$ matrix $A$ is invertible. Then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

PROOF. In view of Propositions 3G and 3C, we have $\det(A)\det(A^{-1}) = \det(I_n) = 1$. The result follows immediately. ○

Finally, the main reason for studying determinants, as outlined in the introduction, is summarized by the following result.

**PROPOSITION 3J.** *Suppose that $A$ is an $n \times n$ matrix. Then $A$ is invertible if and only if* $\det(A) \neq 0$.

PROOF. Suppose that $A$ is invertible. Then $\det(A) \neq 0$ follows immediately from Proposition 3H. Suppose now that $\det(A) \neq 0$. Let us now reduce $A$ by elementary row operations to reduced row echelon form $B$. Then there exist a finite sequence $E_1, \ldots, E_k$ of elementary $n \times n$ matrices such that

$$B = E_k \ldots E_1 A.$$

It follows from Proposition 3G that

$$\det(B) = \det(E_k) \ldots \det(E_1)\det(A).$$

Recall that all elementary matrices are invertible and so have non-zero determinants. It follows that $\det(B) \neq 0$, so that $B$ has no zero rows by Proposition 3B. Since $B$ is an $n \times n$ matrix in reduced row echelon form, it must be $I_n$. We therefore conclude that $A$ is row equivalent to $I_n$. It now follows from Proposition 2N(c) that $A$ is invertible. ◯

Combining Propositions 2Q and 3J, we have the following result.

**PROPOSITION 3K.** *In the notation of Proposition 2N, the following statements are equivalent:*
*(a) The matrix $A$ is invertible.*
*(b) The system $A\mathbf{x} = \mathbf{0}$ of linear equations has only the trivial solution.*
*(c) The matrices $A$ and $I_n$ are row equivalent.*
*(d) The system $A\mathbf{x} = \mathbf{b}$ of linear equations is soluble for every $n \times 1$ matrix $\mathbf{b}$.*
*(e) The determinant $\det(A) \neq 0$.*

### 3.6. Application to Curves and Surfaces

A special case of Proposition 3K states that a homogeneous system of $n$ linear equations in $n$ variables has a non-trivial solution if and only if the determinant if the coefficient matrix is equal to zero. In this section, we shall use this to solve some problems in geometry. We illustrate our ideas by a few simple examples.

EXAMPLE 3.6.1. Suppose that we wish to determine the equation of the unique line on the $xy$-plane that passes through two distinct given points $(x_1, y_1)$ and $(x_2, y_2)$. The equation of a line on the $xy$-plane is of the form $ax + by + c = 0$. Since the two points lie on the line, we must have $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$. Hence

$$\begin{aligned} xa + \ \ yb + c &= 0, \\ x_1a + y_1b + c &= 0, \\ x_2a + y_2b + c &= 0. \end{aligned}$$

Written in matrix notation, we have

$$\begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly there is a non-trivial solution $(a, b, c)$ to this system of linear equations, and so we must have

$$\det \begin{pmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix} = 0,$$

the equation of the line required.

EXAMPLE 3.6.2. Suppose that we wish to determine the equation of the unique circle on the $xy$-plane that passes through three distinct given points $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$, not all lying on a straight line. The equation of a circle on the $xy$-plane is of the form $a(x^2 + y^2) + bx + cy + d = 0$. Since the three points lie on the circle, we must have $a(x_1^2 + y_1^2) + bx_1 + cy_1 + d = 0$, $a(x_2^2 + y_2^2) + bx_2 + cy_2 + d = 0$, and $a(x_3^2 + y_3^2) + bx_3 + cy_3 + d = 0$. Hence

$$\begin{aligned} (x^2 + y^2)a + \ \ xb + \ \ yc + d &= 0, \\ (x_1^2 + y_1^2)a + x_1b + y_1c + d &= 0, \\ (x_2^2 + y_2^2)a + x_2b + y_2c + d &= 0, \\ (x_3^2 + y_3^2)a + x_3b + y_3c + d &= 0. \end{aligned}$$

Written in matrix notation, we have

$$\begin{pmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly there is a non-trivial solution $(a,b,c,d)$ to this system of linear equations, and so we must have

$$\det \begin{pmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{pmatrix} = 0,$$

the equation of the circle required.

Example 3.6.3. Suppose that we wish to determine the equation of the unique plane in 3-space that passes through three distinct given points $(x_1,y_1,z_1)$, $(x_2,y_2,z_2)$ and $(x_3,y_3,z_3)$, not all lying on a straight line. The equation of a plane in 3-space is of the form $ax+by+cz+d=0$. Since the three points lie on the plane, we must have $ax_1+by_1+cz_1+d=0$, $ax_2+by_2+cz_2+d=0$, and $ax_3+by_3+cz_3+d=0$. Hence

$$\begin{aligned} xa+\ \ yb+\ \ zc+d&=0,\\ x_1a+y_1b+z_1c+d&=0,\\ x_2a+y_2b+z_2c+d&=0,\\ x_3a+y_3b+z_3c+d&=0. \end{aligned}$$

Written in matrix notation, we have

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly there is a non-trivial solution $(a,b,c,d)$ to this system of linear equations, and so we must have

$$\det \begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} = 0,$$

the equation of the plane required.

Example 3.6.4. Suppose that we wish to determine the equation of the unique sphere in 3-space that passes through four distinct given points $(x_1,y_1,z_1)$, $(x_2,y_2,z_2)$, $(x_3,y_3,z_3)$ and $(x_4,y_4,z_4)$, not all lying on a plane. The equation of a sphere in 3-space is of the form $a(x^2+y^2+z^2)+bx+cy+dz+e=0$. Since the four points lie on the sphere, we must have

$$\begin{aligned} a(x_1^2+y_1^2+z_1^2)+bx_1+cy_1+dz_1+e&=0,\\ a(x_2^2+y_2^2+z_2^2)+bx_2+cy_2+dz_2+e&=0,\\ a(x_3^2+y_3^2+z_3^2)+bx_3+cy_3+dz_3+e&=0,\\ a(x_4^2+y_4^2+z_4^2)+bx_4+cy_4+dz_4+e&=0. \end{aligned}$$

Hence

$$\begin{aligned} (x^2+y^2+z^2)a+\ \ xb+\ \ yc+\ \ zd+e&=0,\\ (x_1^2+y_1^2+z_1^2)a+x_1b+y_1c+z_1d+e&=0,\\ (x_2^2+y_2^2+z_2^2)a+x_2b+y_2c+z_2d+e&=0,\\ (x_3^2+y_3^2+z_3^2)a+x_3b+y_3c+z_3d+e&=0,\\ (x_4^2+y_4^2+z_4^2)a+x_4b+y_4c+z_4d+e&=0. \end{aligned}$$

Written in matrix notation, we have

$$\begin{pmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly there is a non-trivial solution $(a,b,c,d,e)$ to this system of linear equations, and so we must have

$$\det \begin{pmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{pmatrix} = 0,$$

the equation of the sphere required.

## 3.7. Some Useful Formulas

In this section, we shall discuss two very useful formulas which involve determinants only. The first one enables us to find the inverse of a matrix, while the second one enables us to solve a system of linear equations. The interested reader is referred to Section 3.8 for proofs.

Recall first of all that for any $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix},$$

the number $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is called the cofactor of the entry $a_{ij}$, and the $(n-1)\times(n-1)$ matrix

$$A_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1(j-1)} & \bullet & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \dots & a_{(i-1)(j-1)} & \bullet & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ \bullet & \dots & \bullet & \bullet & \bullet & \dots & \bullet \\ a_{(i+1)1} & \dots & a_{(i+1)(j-1)} & \bullet & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n(j-1)} & \bullet & a_{n(j+1)} & \dots & a_{nn} \end{pmatrix}$$

is obtained from $A$ by deleting row $i$ and column $j$; here $\bullet$ denotes that the entry has been deleted.

DEFINITION. The $n \times n$ matrix

$$\mathrm{adj}(A) = \begin{pmatrix} C_{11} & \dots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \dots & C_{nn} \end{pmatrix}$$

is called the adjoint of the matrix $A$.

REMARK. Note that $\mathrm{adj}(A)$ is obtained from the matrix $A$ first by replacing each entry of $A$ by its cofactor and then by transposing the resulting matrix.

**PROPOSITION 3L.** *Suppose that the $n \times n$ matrix $A$ is invertible. Then*

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

EXAMPLE 3.7.1. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}.$$

Then

$$\operatorname{adj}(A) = \begin{pmatrix} \det\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} & -\det\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} & \det\begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \\ -\det\begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix} & \det\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} & -\det\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \\ \det\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} & -\det\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} & \det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 3 & -2 \\ 4 & 3 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

On the other hand, adding 1 times column 1 to column 2 and then using cofactor expansion on row 1, we have

$$\det(A) = \det\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = -1.$$

It follows that

$$A^{-1} = \begin{pmatrix} -3 & -3 & 2 \\ -4 & -3 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

Next, we turn our attention to systems of $n$ linear equations in $n$ unknowns, of the form

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1, \\ &\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n, \end{aligned}$$

represented in matrix notation in the form

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \tag{8}$$

represent the coefficients and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \tag{9}$$

represents the variables.

For every $j = 1, \ldots, k$, write

$$A_j(\mathbf{b}) = \begin{pmatrix} a_{11} & \ldots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \ldots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{n(j-1)} & b_n & a_{n(j+1)} & \ldots & a_{nn} \end{pmatrix}; \tag{10}$$

in other words, we replace column $j$ of the matrix $A$ by the column $\mathbf{b}$.

**PROPOSITION 3M.** (CRAMER'S RULE) *Suppose that the matrix $A$ is invertible. Then the unique solution of the system $A\mathbf{x} = \mathbf{b}$, where $A$, $\mathbf{x}$ and $\mathbf{b}$ are given by* (8) *and* (9), *is given by*

$$x_1 = \frac{\det(A_1(\mathbf{b}))}{\det(A)}, \qquad \ldots, \qquad x_n = \frac{\det(A_n(\mathbf{b}))}{\det(A)},$$

*where the matrices $A_1(\mathbf{b}), \ldots, A_1(\mathbf{b})$ are defined by* (10).

EXAMPLE 3.7.2. Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Recall that $\det(A) = -1$. By Cramer's rule, we have

$$x_1 = \frac{\det\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 3 & 0 & 3 \end{pmatrix}}{\det(A)} = -3, \qquad x_2 = \frac{\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}}{\det(A)} = -4, \qquad x_3 = \frac{\det\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}}{\det(A)} = 3.$$

Let us check our calculations. Recall from Example 3.7.1 that

$$A^{-1} = \begin{pmatrix} -3 & -3 & 2 \\ -4 & -3 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

We therefore have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 & -3 & 2 \\ -4 & -3 & 2 \\ 2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix}.$$

## 3.8. Further Discussion

In this section, we shall first discuss a definition of the determinant in terms of permutations. In order to do so, we need to make a digression and discuss first the rudiments of permutations on non-empty finite sets.

DEFINITION. Let $X$ be a non-empty finite set. A permutation $\phi$ on $X$ is a function $\phi : X \to X$ which is one-to-one and onto. If $x \in X$, we denote by $x\phi$ the image of $x$ under the permutation $\phi$.

It is not difficult to see that if $\phi : X \to X$ and $\psi : X \to X$ are both permutations on $X$, then $\phi\psi : X \to X$, defined by $x\phi\psi = (x\phi)\psi$ for every $x \in X$ so that $\phi$ is followed by $\psi$, is also a permutation on $X$.

REMARK. Note that we use the notation $x\phi$ instead of our usual notation $\phi(x)$ to denote the image of $x$ under $\phi$. Note also that we write $\phi\psi$ to denote the composition $\psi \circ \phi$. We shall do this only for permutations. The reasons will become a little clearer later in the discussion.

Since the set $X$ is non-empty and finite, we may assume, without loss of generality, that it is $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. We now let $S_n$ denote the set of all permutations on the set $\{1, 2, \ldots, n\}$. In other words, $S_n$ denotes the collection of all functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ that are both one-to-one and onto.

**PROPOSITION 3N.** *For every $n \in \mathbb{N}$, the set $S_n$ has $n!$ elements.*

PROOF. There are $n$ choices for $1\phi$. For each such choice, there are $(n-1)$ choices left for $2\phi$. And so on. ◯

To represent particular elements of $S_n$, there are various notations. For example, we can use the notation

$$\begin{pmatrix} 1 & 2 & \ldots & n \\ 1\phi & 2\phi & \ldots & n\phi \end{pmatrix}$$

to denote the permutation $\phi$.

EXAMPLE 3.8.1. In $S_4$,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$$

denotes the permutation $\phi$, where $1\phi = 2$, $2\phi = 4$, $3\phi = 1$ and $4\phi = 3$. On the other hand, the reader can easily check that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

A more convenient way is to use the cycle notation. The permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$$

can be represented respectively by the cycles $(1\ 2\ 4\ 3)$ and $(1\ 3\ 4)$. Here the cycle $(1\ 2\ 4\ 3)$ gives the information $1\phi = 2$, $2\phi = 4$, $4\phi = 3$ and $3\phi = 1$. Note also that in the latter case, since the image of 2 is 2, it is not necessary to include this in the cycle. Furthermore, the information

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

can be represented in cycle notation by $(1\ 2\ 4\ 3)(1\ 3\ 4) = (1\ 2)$. We also say that the cycles $(1\ 2\ 4\ 3)$, $(1\ 3\ 4)$ and $(1\ 2)$ have lengths 4, 3 and 2 respectively.

EXAMPLE 3.8.2. In $S_6$, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

can be represented in cycle notation as $(1\ 2\ 4\ 3)(5\ 6)$.

EXAMPLE 3.8.3. In $S_4$ or $S_6$, we have $(1\ 2\ 4\ 3) = (1\ 2)(1\ 4)(1\ 3)$.

The last example motivates the following important idea.

Definition. Suppose that $n \in \mathbb{N}$. A permutation in $S_n$ that interchanges two numbers among the elements of $\{1, 2, \ldots, n\}$ and leaves all the others unchanged is called a transposition.

Remark. It is obvious that a transposition can be represented by a 2-cycle, and is its own inverse.

Definition. Two cycles $(x_1\ x_2\ \ldots\ x_k)$ and $(y_1\ y_2\ \ldots\ y_l)$ in $S_n$ are said to be disjoint if the elements $x_1, \ldots, x_k, y_1, \ldots, y_l$ are all different.

The interested reader may try to prove the following result.

**PROPOSITION 3P.** *Suppose that $n \in \mathbb{N}$.*
*(a) Every permutation in $S_n$ can be written as a product of disjoint cycles.*
*(b) For every subset $\{x_1, x_2, \ldots, x_k\}$ of the set $\{1, 2, \ldots, n\}$, where the elements $x_1, x_2, \ldots, x_k$ are distinct, the cycle $(x_1\ x_2\ \ldots\ x_k)$ satisfies*

$$(x_1\ x_2\ \ldots\ x_k) = (x_1\ x_2)(x_1\ x_3)\ldots(x_1\ x_k);$$

*in other words, every cycle can be written as a product of transpositions.*
*(c) Consequently, every permutation in $S_n$ can be written as a product of transpositions.*

Example 3.8.4. In $S_9$, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 2 & 5 & 1 & 7 & 8 & 4 & 9 & 6 \end{pmatrix}$$

can be written in cycle notation as $(1\ 3\ 5\ 7\ 4)(6\ 8\ 9)$. By Theorem 3P(b), we have

$$(1\ 3\ 5\ 7\ 4) = (1\ 3)(1\ 5)(1\ 7)(1\ 4) \qquad \text{and} \qquad (6\ 8\ 9) = (6\ 8)(6\ 9).$$

Hence the permutation can be represented by $(1\ 3)(1\ 5)(1\ 7)(1\ 4)(6\ 8)(6\ 9)$.

Definition. Suppose that $n \in \mathbb{N}$. Then a permutation in $S_n$ is said to be even if it is representable as the product of an even number of transpositions and odd if it is representable as the product of an odd number of transpositions. Furthermore, we write

$$\epsilon(\phi) = \begin{cases} +1 & \text{if } \phi \text{ is even,} \\ -1 & \text{if } \phi \text{ is odd.} \end{cases}$$

Remark. It can be shown that no permutation can be simultaneously odd and even.

We are now in a position to define the determinant of a matrix. Suppose that

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \tag{11}$$

is an $n \times n$ matrix.

Definition. By an elementary product from the matrix $A$, we mean the product of $n$ entries of $A$, no two of which are from the same row or same column.

It follows that any such elementary product must be of the form

$$a_{1(1\phi)}a_{2(2\phi)}\ldots a_{n(n\phi)},$$

where $\phi$ is a permutation in $S_n$.

DEFINITION. By the determinant of an $n \times n$ matrix $A$ of the form (11), we mean the sum

$$\det(A) = \sum_{\phi \in S_n} \epsilon(\phi) a_{1(1\phi)} a_{2(2\phi)} \ldots a_{n(n\phi)}, \tag{12}$$

where the summation is over all the $n!$ permutations $\phi$ in $S_n$.

It is be shown that the determinant defined in this way is the same as that defined earlier by row or column expansions. Indeed, one can use (12) to establish Proposition 3A. The very interested reader may wish to make an attempt. Here we confine our study to the special cases when $n = 2$ and $n = 3$. In the two examples below, we use $e$ to denote the identity permutation.

EXAMPLE 3.8.5. Suppose that $n = 2$. We have the following:

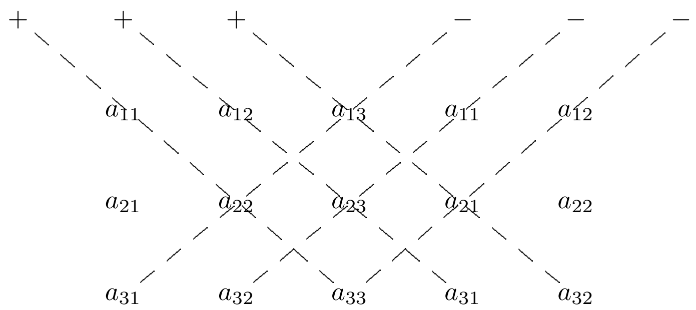
| elementary product | permutation | sign | contribution |
|---|---|---|---|
| $a_{11}a_{22}$ | $e$ | $+1$ | $+a_{11}a_{22}$ |
| $a_{12}a_{21}$ | $(1\ 2)$ | $-1$ | $-a_{12}a_{21}$ |

Hence $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ as shown before.

EXAMPLE 3.8.6. Suppose that $n = 3$. We have the following:

| elementary product | permutation | sign | contribution |
|---|---|---|---|
| $a_{11}a_{22}a_{33}$ | $e$ | $+1$ | $+a_{11}a_{22}a_{33}$ |
| $a_{12}a_{23}a_{31}$ | $(1\ 2\ 3)$ | $+1$ | $+a_{12}a_{23}a_{31}$ |
| $a_{13}a_{21}a_{32}$ | $(1\ 3\ 2)$ | $+1$ | $+a_{13}a_{21}a_{32}$ |
| $a_{13}a_{22}a_{31}$ | $(1\ 3)$ | $-1$ | $-a_{13}a_{22}a_{31}$ |
| $a_{11}a_{23}a_{32}$ | $(2\ 3)$ | $-1$ | $-a_{11}a_{23}a_{32}$ |
| $a_{12}a_{21}a_{33}$ | $(1\ 2)$ | $-1$ | $-a_{12}a_{21}a_{33}$ |

Hence $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$. We have the picture below:




Next, we discuss briefly how one may prove Proposition 3G concerning the determinant of the product of two matrices. The idea is to use elementary matrices. Corresponding to Proposition 3D, we can easily establish the following result.

**PROPOSITION 3Q.** *Suppose that $E$ is an elementary matrix.*
*(a) If $E$ arises from interchanging two rows of $I_n$, then* $\det(E) = -1$.
*(b) If $E$ arises from adding one row of $I_n$ to another row, then* $\det(E) = 1$.
*(c) If $E$ arises from multiplying one row of $I_n$ by a non-zero constant $c$, then* $\det(E) = c$.

Combining Propositions 3D and 3Q, we can establish the following intermediate result.

**PROPOSITION 3R.** *Suppose that $E$ is an $n \times n$ elementary matrix. Then for any $n \times n$ matrix $B$, we have* $\det(EB) = \det(E)\det(B)$.

Proof of Proposition 3G. Let us reduce $A$ by elementary row operations to reduced row echelon form $A'$. Then there exist a finite sequence $G_1, \ldots, G_k$ of elementary matrices such that $A' = G_k \ldots G_1 A$. Since elementary matrices are invertible with elementary inverse matrices, it follows that there exist a finite sequence $E_1, \ldots, E_k$ of elementary matrices such that

$$A = E_1 \ldots E_k A'. \tag{13}$$

Suppose first of all that $\det(A) = 0$. Then it follows from (13) that the matrix $A'$ must have a zero row. Hence $A'B$ must have a zero row, and so $\det(A'B) = 0$. But $AB = E_1 \ldots E_k(A'B)$, so it follows from Proposition 3R that $\det(AB) = 0$. Suppose next that $\det(A) \neq 0$. Then $A' = I_n$, and so it follows from (13) that $AB = E_1 \ldots E_k B$. The result now follows on applying Proposition 3R. ○

We complete this chapter by establishing the two formulas discussed in Section 3.7.

Proof of Proposition 3L. It suffices to show that

$$A\,\mathrm{adj}(A) = \det(A) I_n, \tag{14}$$

as this clearly implies

$$A\left(\frac{1}{\det(A)}\mathrm{adj}(A)\right) = I_n,$$

giving the result. To show (14), note that

$$A\,\mathrm{adj}(A) = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & \ldots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \ldots & C_{nn} \end{pmatrix}. \tag{15}$$

Suppose that the right hand side of (15) is equal to

$$B = \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{pmatrix}.$$

Then for every $i, j = 1, \ldots, n$, we have

$$b_{ij} = a_{i1}C_{j1} + \ldots + a_{in}C_{jn}. \tag{16}$$

It follows that when $i = j$, we have

$$b_{ii} = a_{i1}C_{i1} + \ldots + a_{in}C_{in} = \det(A).$$

On the other hand, if $i \neq j$, then (16) is equal to the determinant of the matrix obtained from $A$ by replacing row $j$ by row $i$. This matrix has therefore two identical rows, and so the determinant is 0 (why?). The identity (14) follows immediately. ○

Proof of Proposition 3M. Since $A$ is invertible, it follows from Proposition 3L that

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A).$$

By Proposition 2P, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\text{adj}(A)\mathbf{b}.$$

Written in full, this becomes

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & \ldots & C_{n1} \\ \vdots & & \vdots \\ C_{1n} & \ldots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} b_1C_{11} + \ldots + b_nC_{n1} \\ \vdots \\ b_1C_{1n} + \ldots + b_nC_{nn} \end{pmatrix}.$$

Hence, for every $j = 1, \ldots, n$, we have

$$x_j = \frac{b_1C_{1j} + \ldots + b_nC_{nj}}{\det(A)}.$$

To complete the proof, it remains to show that

$$b_1C_{1j} + \ldots + b_nC_{nj} = \det(A_j(\mathbf{b})).$$

Note, on using cofactor expansion by column $j$, that

$$\begin{aligned}\det(A_j(\mathbf{b})) &= \sum_{i=1}^{n} b_i(-1)^{i+j} \det \begin{pmatrix} a_{11} & \ldots & a_{1(j-1)} & \bullet & a_{1(j+1)} & \ldots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & \ldots & a_{(i-1)(j-1)} & \bullet & a_{(i-1)(j+1)} & \ldots & a_{(i-1)n} \\ \bullet & \ldots & \bullet & \bullet & \bullet & \ldots & \bullet \\ a_{(i+1)1} & \ldots & a_{(i+1)(j-1)} & \bullet & a_{(i+1)(j+1)} & \ldots & a_{(i+1)n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \ldots & a_{n(j-1)} & \bullet & a_{n(j+1)} & \ldots & a_{nn} \end{pmatrix} \\ &= \sum_{i=1}^{n} b_i(-1)^{i+j} \det(A_{ij}) = \sum_{i=1}^{n} b_iC_{ij}\end{aligned}$$

as required. ○

## Problems for Chapter 3

1. Compute the determinant of each of the matrices in Problem 2.6.

2. Find the determinant of each of the following matrices:

$$P = \begin{pmatrix} 1 & 3 & 2 \\ 8 & 4 & 0 \\ 2 & 1 & 2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{pmatrix}.$$

3. Find the determinant of the matrix

$$\begin{pmatrix} 3 & 4 & 5 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & 3 & 6 & 3 \\ 7 & 2 & 9 & 4 \end{pmatrix}.$$

4. By using suitable elementary row and column operations as well as row and column expansions, show that

$$\det \begin{pmatrix} 2 & 3 & 7 & 1 & 3 \\ 2 & 3 & 7 & 1 & 5 \\ 2 & 3 & 6 & 1 & 9 \\ 4 & 6 & 2 & 3 & 4 \\ 5 & 8 & 7 & 4 & 5 \end{pmatrix} = 2.$$

[Remark: Note that rows 1 and 2 of the matrix are almost identical.]

5. By using suitable elementary row and column operations as well as row and column expansions, show that

$$\det \begin{pmatrix} 2 & 1 & 5 & 1 & 3 \\ 2 & 1 & 5 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 0 & 1 \\ 2 & 1 & 6 & \pi & 7 \end{pmatrix} = 2.$$

[Remark: The entry $\pi$ is not a misprint!]

6. If $A$ and $B$ are square matrices of the same size and $\det A = 2$ and $\det B = 3$, find $\det(A^2B^{-1})$.

7. a) Compute the Vandermonde determinants

$$\det \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \qquad \text{and} \qquad \det \begin{pmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{pmatrix}.$$

b) Establish a formula for the Vandermonde determinant

$$\det \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix}.$$

8. Compute the determinant

$$\det\begin{pmatrix} a & b & c \\ a+x & b+x & c+x \\ a+y & b+y & c+y \end{pmatrix}.$$

9. For each of the matrices below, compute its adjoint and use Proposition 3L to calculate its inverse:

a) $\begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 3 & 5 & 4 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

10. Use Cramer's rule to solve the system of linear equations

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 4, \\ -x_1 \qquad + 2x_3 &= 2, \\ 3x_1 + x_2 + 3x_3 &= -2. \end{aligned}$$