Chapter 7

Eigenvalues and Eigenvectors

7.1 Eigenvalues and Eigenvectors

Homework: [Textbook, §7.1 Ex. 5, 11, 15, 19, 25, 27, 61, 63, 65].

Optional Homework:[Textbook, §7.1 Ex. 53, 59].

In this section, *we introduce eigenvalues and eigenvectors. This is one of most fundamental and most useful concepts in linear algebra.*

Definition 7.1.1 Let A be an $n \times n$ matrix. A scalar λ is said to be a **eigenvalue** of A, if

$$A\mathbf{x} = \lambda\mathbf{x} \quad for \quad some \quad vector \quad \mathbf{x} \neq \mathbf{0}.$$

The vector $\mathbf{x}$ is called an **eigenvector** corresponding to λ. *The zero vector* $\mathbf{0}$ *is never an eigenvectors, by definition.*

Reading assignment: Read [Textbook, Examples 1, 2, page 423].

7.1.1 Eigenspaces

Given a square matrix A, there will be many eigenvectors corresponding to a given eigenvalue λ. In fact, together with the zero vector $\mathbf{0}$, the set of all eigenvectors corresponding to a given eigenvalue λ will form a subspace. We state the same as a theorem:

Theorem 7.1.2 Let A be an $n \times n$ matrix and λ is an eigenvalue of A. Then the set

$$E(\lambda) = \{\mathbf{0}\} \cup \{\mathbf{x} : \mathbf{x} \quad is \quad an \quad eigenvector \quad corresponding \quad to \quad \lambda\}$$

(*of all eigenvalues corresponding to* λ, *together with* $\mathbf{0}$) is a subspace of $\mathbb{R}^n$. This subspace $E(\lambda)$ is called the **eigenspace** of λ.

Proof. Since $\mathbf{0} \in E(\lambda)$, we have $E(\lambda)$ is nonempty. Because of theorem 4.3.3, we need only to check that $E(\lambda)$ is closed under addition and scalar multiplication. Suppose $\mathbf{x}, \mathbf{y} \in E(\lambda)$ and c be a scalar. Then,

$$A\mathbf{x} = \lambda\mathbf{x} \quad and \quad A\mathbf{y} = \lambda\mathbf{y}.$$

So,

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x} + \mathbf{y}).$$

So, $\mathbf{x}+\mathbf{y}$ is an eigenvalue corresponding to λ or zero. So, $\mathbf{x}+\mathbf{y} \in E(\lambda)$ and $E(\lambda)$ is closed under addition. Also,

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x}).$$

So, $c\mathbf{x} \in E((\lambda)$ and $E(\lambda)$ is closed under scalar multiplication. Therefore, $E(\lambda)$ is a subspace of $\mathbb{R}^n$. The proof is complete. ■

Reading assignment: Read [Textbook, Examples 3, page 423].

Theorem 7.1.3 Let A be a square matrix of size $n \times n$. Then

1. Then a scalar λ is an eigenvalue of A if and only if

$$\det(\lambda I - A) = 0,$$

   here I denotes the identity matrix.

2. A vector $\mathbf{x}$ is an eigenvector, of A, corresponding to λ if and only if $\mathbf{x}$ is a nozero solution

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

Proof. By definition, λ is an eigenvalue of A if and only if, for some nonzero $\mathbf{x}$, we have

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x} \Leftrightarrow (\lambda I - A)\mathbf{x} = \mathbf{0} \Leftrightarrow \det(\lambda I - A) = 0.$$

The last equivalence is given by [Textbook, §3.3], which we did not cover. This establishes (1) of the theorem. The proof of (2) is obvious or same as that of (1). This completes the proof. ■

Definition 7.1.4 Let A be a square matrix of size $n \times n$. Then the equation

$$\det(\lambda I - A) = 0$$

is called the **characteristic equation** of A. (*The German word 'eigen' roughly means 'characteristic'.*)

1. Using induction and expanding det$((\lambda I - A)$, it follows that

$$\det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

which is a polynomial in λ, of degree n. This polynomial is called the **characteristic polynomial** of A.

2. If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

then

$$(\lambda I - A) = \begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & \lambda - a_{33} & \cdots & -a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & \lambda - a_{nn} \end{bmatrix}.$$

So, the characteristic polynomial is the determinant of this matrix.

Method of finding eigenvalues and eigenvectors is as follows: Let A be an $n \times n$ matrix.

1. To find the eigenvalues of A solve the characteristic equation

$$\det(\lambda I - A) = 0.$$

This is a polynomial equation in λ of degree n. We only consider real roots of this equation, in this class.

2. Given an eigenvalue λ_i (i.e. *a root of the characteristic equation*), to find the eigenspace $E(\lambda_i)$, corresponding to λ_i, we solve the linear system
$$(\lambda_i I - A)\mathbf{x} = \mathbf{0}.$$
As usual, to solve this we reduce it to the row echelon form or Gauss-Jordan form. Since λ_i is an eigenvalue, at least one row of the echlon form will be zero.

Reading assignment: Read [Textbook, Examples 4-7, page 426-].

Exercise 7.1.5 (Ex. 6, p. 432) Let
$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

1. Verify that $\lambda_1 = 5$ is a eigenvalue of A and $\mathbf{x_1} = (1, 2, -1)^T$ is a corresponding eigenvector.

   Solution: We need to check $A\mathbf{x_1} = 5\mathbf{x_1}$, We have
$$A\mathbf{x_1} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 5\mathbf{x_1}.$$
   So, assertion is verified.

2. Verify that $\lambda_2 = -3$ is a eigenvalue of A and $\mathbf{x_2} = (-2, 1, 0)^T$ is a corresponding eigenvector.

   Solution: We need to check $A\mathbf{x_2} = -3\mathbf{x_2}$, We have
$$A\mathbf{x_2} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = -3\mathbf{x_2}.$$
   So, assertion is verified.

3. Verify that $\lambda_3 = -3$ is a eigenvalue of A and $\mathbf{x_3} = (3, 0, 1)^T$ is a corresponding eigenvector.

   Solution: We need to check $A\mathbf{x_3} = -3\mathbf{x_3}$, We have

$$A\mathbf{x_3} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ -3 \end{bmatrix} = -3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = -3\mathbf{x_3}.$$

   So, assertion is verified.

Exercise 7.1.6 (Ex. 14, p. 433) Let

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix}.$$

1. Determine whether $\mathbf{x} = (1, 1, 0)^T$ is an eigenvector of A.

   Solution: We have

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

   for all λ. So, $\mathbf{x}$ is not an eigenvector of A.

2. Determine whether $\mathbf{x} = (-5, 2, 1)^T$ is an eigenvector of A.

   Solution: We have

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix} = 0\mathbf{x}..$$

   So, $\mathbf{x}$ is an eigenvector and corresponding eigenvalue is $\lambda = 0$.

3. Determine whether $\mathbf{x} = (0, 0, 0)^T$ is an eigenvector of A.

   Solution: No, $\mathbf{0}$ is, by definition, never an eigenvector.

4. Determine whether $\mathbf{x} = (2\sqrt{6} - 3, -2\sqrt{6} + 6, 3)^T$ is an eigenvector of A.

   Solution: We have

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 5 \\ 0 & -2 & 4 \\ 1 & -2 & 9 \end{bmatrix} \begin{bmatrix} 2\sqrt{6} - 3 \\ -2\sqrt{6} + 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\sqrt{6} + 12 \\ 4\sqrt{6} \\ 6\sqrt{6} + 12 \end{bmatrix} \neq \lambda \begin{bmatrix} 2\sqrt{6} - 3 \\ -2\sqrt{6} + 6 \\ 3 \end{bmatrix}.$$

   So, $\mathbf{x}$ is not an eigenvector of A.

Exercise 7.1.7 (Ex. 20, p. 433) Let

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}.$$

1. Find the characteristic equation of A.

   Solution: The characteristic polynomial is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda + 5 & 0 & 0 \\ -3 & \lambda - 7 & 0 \\ -4 & 2 & \lambda - 3 \end{vmatrix} = (\lambda + 5)(\lambda - 7)(\lambda - 3).$$

   So, the characteristic equation is

$$(\lambda + 5)(\lambda - 7)(\lambda - 3) = 0.$$

2. Find eigenvalues (and corresponding eigenvectors) of A.

   Solution: Solving the characteristic equation, the eigenvalues are $\lambda = -5, 7, 3$.

(a) To find an eigenvector corresponding to $\lambda = -5$, wehave to solve $(-5I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & -12 & 0 \\ -4 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving, we get

$$x = -\frac{16}{9}t \qquad y = \frac{4}{9}t \qquad z = t.$$

So, that eigenspace of $\lambda = -5$ is

$$\left\{ \left(-\frac{16}{9}t, \frac{4}{9}t, t \right) : t \in \mathbb{R} \right\}.$$

In particular, with $t = 1$, an eigenvector, for eigenvalue $\lambda = -5$, is $\left(-\frac{16}{9}, \frac{4}{9}, 1\right)^T$.

(b) To find an eigenvector corresponding to $\lambda = 7$, wehave to solve $(7I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 12 & 0 & 0 \\ -3 & 0 & 0 \\ -4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving, we get

$$x = 0 \qquad y = -2t \qquad z = t.$$

So, that eigenspace of $\lambda = 7$ is

$$\{(0, -2t, t) : t \in \mathbb{R}\}.$$

In particular, with $t = 1$, an eigenvector, for eigenvalue $\lambda = 7$, is $(0, -2, 1)^T$.

(c) To find an eigenvector corresponding to $\lambda = 3$, wehave to solve $(3I - A)\mathbf{x} = \mathbf{0}$ or

$$\begin{bmatrix} 8 & 0 & 0 \\ -3 & -4 & 0 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving, we get

$$x = 0 \qquad y = 0 \qquad z = t.$$

So, that eigenspace of $\lambda = 3$ is

$$\{(0, 0, t) : t \in \mathbb{R}\}.$$

In particular, with $t = 1$, an eigenvector, for eigenvalue $\lambda = 3$, is $(0, 0, 1)^T$.

Exercise 7.1.8 (Ex. 66, p. 435) Let

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Find the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 3$.

Solution: The eigenspace $E(3)$ is the solution space of the system $(3I - A)\mathbf{x} = \mathbf{x}$, or

$$\begin{bmatrix} 3-3 & -1 & -1 \\ 0 & 3-3 & -1 \\ 0 & 0 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix

$$C = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

has rank 2. Since

$$rank(C) + nullity(C) = 3, \quad we \quad nullity(C) = 1.$$

Therefore, $\dim E(3) = 1$.

7.2 Diagonalization

Homework: [Textbook, Ex. 1, 3, 5, 9, 11, 13, 17, 19; p.444].

In this section, *we discuss, given a square matrix A, when or whether we can find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. This problem is closely associated to eigenvalues and eigenvectors.*

First, we recall the definition 6.4.1, as follows:

Definition 7.2.1 Suppose A, B are two square matrices of size $n \times n$. We say A, B are **similar,** if $A = P^{-1}BP$ for some invertible matrix P.

We also define the following:

Definition 7.2.2 Suppose A is a square matrix of size $n \times n$. We say that A is **diagonalizable,** if there exists a invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

So, our question is which matrices are diagonalizable? Following theorem has some answer.

Theorem 7.2.3 Suppose A is a square matrix of size $n \times n$. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof. Suppose A is diagonalizable. So, there is an invertible matrix

P such that $P^{-1}AP = D$ is a diagonal matrix. Write

$$P = \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{bmatrix} \quad and \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

where $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are the cllumns of P. We have $AP = PD$. So,

$$A \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{bmatrix} = \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Therefore, $i =, 2, \ldots, n$ we have $A\mathbf{p_i} = \lambda_i \mathbf{p_i}$ and so $\mathbf{p_i}$ are eigenvectors of A. Also, since P is invertible $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ are linearly independent. So, A have n linearly independent eigenvectors.

To prove the converse, assume A has n linearly independent eigenvectors. Let $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$ be n linearly independent eigenvectors of A. Then Then, for $i =, 2, \ldots, n$ we have, $A\mathbf{p_i} = \lambda_i \mathbf{p_i}$ for some λ_i. Write,

$$P = \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{bmatrix} \quad and \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

It follows easily that $AP = PD$. Since, columns af P are linearly independent, it follows that P is invertible. Therefore, $P^{-1}AP = D$ is a diagonal matrix. So, the proof is complete. ■

Steps for Diagonalizing an $n \times n$ matrix:

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\mathbf{p_1}, \mathbf{p_2}, \cdots, \mathbf{p_n}$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n independent eigenvectors do not exists, then A is not diagonalizable.

2. If A has n linearly independent eigenvectors as above, write

$$P = \begin{bmatrix} \mathbf{p_1} & \mathbf{p_2} & \cdots & \mathbf{p_n} \end{bmatrix} \quad and \quad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

3. Then $D = P^{-1}AP$ is a diagonal matrix.

Theorem 7.2.4 Suppose A is an $n \times n$ matrix. If A has n distinct eigenvalues, then the corresponding eignevectors are linearly independent and A is diagonizable.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct eigenvalues of A and let $\mathbf{x_i}$ eigenvectosr corresponding to λ_i. So, $A\mathbf{x_i} = \lambda_i \mathbf{x_i}$.

We claim that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are linearly independent. If not, assume form some $m < n$, we have $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ are mutually linearly independent and $\mathbf{x_{m_1}}$ is in $Span\left(\{\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}\}\right)$. So, we can write

$$\mathbf{x_{m+1}} = c_1\mathbf{x_1} + c_2\mathbf{x_2} + \cdots + c_m\mathbf{x_m} \qquad Eqn - I.$$

Here, at least one $c_i \neq 0$. Multiply by A and use the equation $A\mathbf{x_i} = \lambda_i \mathbf{x_i}$, we have

$$\lambda_{m+1}\mathbf{x_{m+1}} = \lambda_1 c_1\mathbf{x_1} + \lambda_2 c_2\mathbf{x_2} + \cdots + \lambda_m c_m\mathbf{x_m} \qquad Eqn - II.$$

Multiply Eqn-I by λ_{m+1}, we have

$$\lambda_{m+1}\mathbf{x_{m+1}} = \lambda_{m+1} c_1\mathbf{x_1} + \lambda_{m+1} c_2\mathbf{x_2} + \cdots + \lambda_{m+1} c_m\mathbf{x_m} \qquad Eqn - III.$$

Subtract Eqn-II from Eqn -III:

$$(\lambda_{m+1} - \lambda_1)c_1\mathbf{x_1} + (\lambda_{m+1} - \lambda_2)c_2\mathbf{x_2} + \cdots + (\lambda_{m+1} - \lambda_m)c_m\mathbf{x_m} = \mathbf{0}.$$

Since, at least one $c_i \neq 0$. and since λ_i are distinct, at least one coefficient $(\lambda_{m+1} - \lambda_i)c_i \neq 0$. This contrdicts that $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_m}$ are mutually linearly independent. So, it is established that these n eigenvectors

$\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ are mutually linearly independent. (**This method of proof is called a proof by contrapositive argument.**) Now, by theorem 7.2.3, A is diagonizable. So, the proof is complete. ■

Reading assignment: Read [Textbook, Examples 1-7, page 436-].

Exercise 7.2.5 (Ex. 6, p. 444) Let

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \textit{and} \quad P = \begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Verify that A is diagonalizable, by computing $P^{-1}AP$.

Solution: We do it in a two steps.

1. Use TI to compute

$$P^{-1} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & .5 \\ 0 & 0 & .5 \end{bmatrix}.$$

2. Use TI to compute

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

So, it is verified that $P^{-1}AP$ is a diagonal matrix.

Exercise 7.2.6 (Ex. 10, p. 444) Let

$$A = \begin{bmatrix} 1 & .5 \\ -2 & -1 \end{bmatrix}.$$

Show that A is not diagonalizable.

Solution: To do this, we have find and count the dimensions of all the eigenspaces $E(\lambda)$. We do it in a few steps.

1. First, find all the eigenvalues. To do this, we solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -.5 \\ 2 & \lambda + 1 \end{vmatrix} = \lambda^2 = 0.$$

   So, $\lambda = 0$ is the only eigenvalue of A.

2. Now we compute the eigenspace $E(0)$ of the eigenvalue $\lambda = 0$. We have $E(0)$ is solution space of

$$(0I - A)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & .5 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Using TI (or by hand), a parametric solution of this system is given by

$$x = -.5t \quad y = t. \quad so \quad E(0) = \{(-.5t, t) : t \in \mathbb{R}\} = \mathbb{R}(-.5, 1).$$

   So, the (sum of) dimension(s) of the eigenspace(s)

$$= \dim E(0) = 1 < 2.$$

   Therefore A is not diagonizable.

Exercise 7.2.7 (Ex. 14, p. 444) Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

Show that A is not diagonalizable.

Solution: To do this, we have find and count the dimensions of all the eigenspaces $E(\lambda)$. We do it in a few steps.

1. First, find all the eigenvalues. To do this, we solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 & 1 \\ 0 & \lambda + 1 & -2 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 1)^2 = 0.$$

   So, $\lambda = -1, 2$ are the only eigenvalues of A.

2. Now we compute the dimension $\dim E(-1)$ of the eigenspace $E(-1)$ of the eigenvalue $\lambda = -1$. We have $E(-1)$ is solution space of

$$(-I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad or \quad \begin{bmatrix} -3 & -1 & +1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   (*We will avoid solving this system.*) The rank of the coefficient matrix is 2. So,

$$\dim(E(-1)) = nullity = 3 - rank = 3 - 2 = 1.$$

3. Now we compute the dimension $\dim E(2)$ of eigenspace $E(2)$ of the eigenvalue $\lambda = 2$. We have $E(2)$ is solution space of

$$(-I - A)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad or \quad \begin{bmatrix} 0 & -1 & +1 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   Use TI (or look at the columns) to see that rank of the coefficient matrix is 2. So,

$$\dim(E(2)) = nullity = 3 - rank = 3 - 2 = 1.$$

4. So, the sum of dimensions of the eigenspaces

$$= \dim E(-1) + \dim E(2) = 2 < 3.$$

   Therefore A is not diagonizable.

Exercise 7.2.8 (Ex. 20, p. 444) Let

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

Find the eigenvalues of A and determine whether there is a sufficient number of them to guarantee that A is diagonalizable.

Solution: First, find all the eigenvalues. To do this, we solve

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 4 & -3 & 2 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda - 1)(\lambda + 2) = 0.$$

So, $\lambda = 4, 1, -2$ are the eigenvalues of A. This means, A has three distinct eigenvalues. Therefore, by theorem 7.2.4, A is diagonalizable.

Bibliography

[Textbook] Ron Larson and David C. Falvo, *Elementary Linear Algebra,* Houghton Miffin