1.5 Elementary Matrices

1.5.1 Definitions and Examples

The transformations we perform on a system or on the corresponding augmented matrix, when we attempt to solve the system, can be simulated by matrix multiplication. More precisely, each of the three transformations we perform on the augmented matrix can be achieved by multiplying the matrix on the left (pre-multiply) by the correct matrix. The correct matrix can be found by applying one of the three elementary row transformation to the identity matrix. Such a matrix is called an elementary matrix. More precisely, we have the following definition:

Definition 95 *An **elementary matrix** is an $n \times n$ matrix which can be obtained from the identity matrix I_n by performing on I_n a single elementary row transformation.*

Example 96 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is an elementary matrix. It can be obtained by switching rows 1 and 2 of the identity matrix. In other words, we are performing on the identity matrix $(R_1) \leftrightarrow (R_2)$.*

Example 97 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *is an elementary matrix. It can be obtained by multiplying row 2 of the identity matrix by 5. In other words, we are performing on the identity matrix $(5R_2) \rightarrow (R_2)$.*

Example 98 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ *is an identity matrix. It can be obtained by replacing row 3 of the identity matrix by row 3 plus -2 times row 1. In other words, we are performing on the identity matrix $(R_3 - 2R_1) \rightarrow (R_3)$.*

Since there are three elementary row transformations, there are three different kind of elementary matrices. They will be described in more details below. Elementary matrices are important because they can be used to simulate the elementary row transformations. If we want to perform an elementary row transformation on a matrix A, it is enough to pre-multiply A by the elementary matrix obtained from the identity by the same transformation. This is illustrated below for each of the three elementary row transformations.

1.5.2 Elementary Matrices and Elementary Row Operations

Interchanging Two Rows $(R_i) \leftrightarrow (R_j)$

Proposition 99 *To interchange rows i and j of matrix A, that is to simulate $(R_i) \leftrightarrow (R_j)$, we can pre-multiply A by the elementary matrix obtained from the* *identity matrix in which rows i and j have been interchanged. In other words, we have performed on the identity matrix the transformation we want to perform on A.*

Example 100 *What should we pre-multiply $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ by if we want to interchange rows 1 and 3?*

We start with the 4×4 identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,we then interchange rows 1 and 3 in it to obtain $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is the desired elementary matrix. We can check that if we pre-multiply A by this matrix, the resulting matrix will be A in which rows 1 and 3 have been interchanged.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Multiplying a Row by a Constant $(mR_i) \leftrightarrow (R_i)$

Proposition 101 *To multiply row i of matrix A by a number m, that is to simulate $(mR_i) \leftrightarrow (R_i)$, we can pre-multiply A by the elementary matrix obtained from the identity matrix in which row i has been multiplied by m. In other words, we have performed on the identity matrix the transformation we want to perform on A.*

Example 102 *What should we pre-multiply $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ by if we want to multiply row 3 by m?*

We start with the 4×4 identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,we then multiply row 3 by m to obtain $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This is the desired elementary matrix. We can check that if we pre-multiply A by this matrix, the resulting matrix will be

A in which row 3 has been multiplied by m.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ma_{31} & ma_{32} & ma_{33} & ma_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Remark 103 *To actually create the matrix which performs $(mR_i) \leftrightarrow (R_i)$, we do not need to perform the same operation on the identity matrix. It would be a waste of time and computations as most of the entries of the identity matrix are 0. We can see that it is enough to do the following:*

1. *Generate the identity matrix of the correct size. Call it $B = (b_{ij})$. Then, we have $b_{ii} = 1 \; \forall i$ and $b_{ij} = 0$ if $i \neq j$.*

2. *Set $b_{ii} = m$ where i is the index of the row affected by the transformation.*

Replacing a Row by Itself Plus a Multiple of Another $(R_j + mR_i) \leftrightarrow (R_j)$

Proposition 104 *To simulate $(R_j + mR_i) \leftrightarrow (R_j)$ on a matrix A, we can pre-multiply A by the elementary matrix obtained from the identity matrix in which the same transformation has been applied.*

Example 105 *What should we pre-multiply* $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ *by if we want to simulate $(R_3 + mR_1) \leftrightarrow (R_3)$?*

We start with the 4×4 identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$*, we then apply $(R_3 + mR_1) \leftrightarrow (R_3)$ to it to obtain* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$*. This is the desired matrix. We can check that if we pre-multiply A by this matrix, the resulting matrix will be A in which $(R_3 + mR_1) \leftrightarrow (R_3)$ has been performed.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ ma_{11} + a_{31} & ma_{12} + a_{32} & ma_{13} + a_{33} & ma_{14} + a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Remark 106 *To actually create the matrix which performs* $(R_j + mR_i) \leftrightarrow (R_j)$*, we do not need to perform the same operation on the identity matrix. It would be a waste of time and computations as most of the entries of the identity matrix are* 0*. We can see that it is enough to do the following:*

1. *Generate the identity matrix of the correct size. Call it* $B = (b_{ij})$*. Then, we have* $b_{ii} = 1\ \forall i$ *and* $b_{ij} = 0$ *if* $i \neq j$*.*
2. *Set* $b_{ji} = m$ *where* j *and* i *are the indexes of the rows affected by the transformation.*

1.5.3 Some Properties of Elementary Matrices

Theorem 107 *The elementary matrices are nonsingular. Furthermore, their inverse is also an elementary matrix. That is, we have:*

1. *The inverse of the elementary matrix which interchanges two rows is itself. For example, the inverse of* $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *is the matrix* $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *as the computation below shows.*
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
2. *The inverse of the elementary matrix which simulates* $(mR_i) \leftrightarrow (R_i)$ *is the elementary matrix which simulates* $\left(\frac{1}{m}R_i\right) \leftrightarrow (R_i)$*. For example, the inverse of* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *is the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *as the computation below shows.*
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
3. *The inverse of the elementary matrix which simulates* $(R_j + mR_i) \leftrightarrow (R_j)$ *is the elementary matrix which simulates* $(R_j - mR_i) \leftrightarrow (R_j)$*. For example, the inverse of the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$*, is the matrix* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Since inverses are unique, the computation below proves it.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -m & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark 108 *To simulate one of the three transformations on a matrix, we pre-multiply the matrix by another matrix which is obtained from the identity matrix by applying the same transformation to it. If instead of pre-multiplying we post-multiply, that is multiply on the right, the transformation would be applied on the columns, not on the rows.*

1.5.4 Gaussian Elimination and Elementary Matrices

When we transform a matrix in row-echelon form using Gaussian elimination, we do it by applying several elementary row operations. Therefore, this can be simulated by using elementary matrices. Rather than explaining how this is done in general; the notation gets complicated. We illustrate the technique with a specific example.

Example 109 *Write the matrix A below in row-echelon form*

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 1 & 5 & 3 \end{bmatrix}$$

The first step in the Gauss transformation is to set to 0 the entries below a_{11}. This is done in two steps. First, we want to set a_{21} to 0. For this, we want to simulate $(R_2 - 2R_1) \rightarrow (R_2)$ therefore, we use the matrix

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with pre-multiplication. You will notice that if we let $A_1 = E_1 A$, then

$$A_1 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 1 & 5 & 3 \end{bmatrix}$$

Then, we want to set a_{31} to 0. For this, we want to simulate $(R_3 - R_1) \rightarrow (R_3)$ therefore, we use the matrix

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

with pre-multiplication. You will notice that if we let $A_2 = E_1 A_1$, *then*

$$A_2 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 2 & -2 \end{bmatrix}$$

We have managed to set to 0 *the entries below* a_{11}. *Next, we set* a_{32} *to* 0. *For this, we want to simulate* $\left(R_3 + \frac{2}{5} R_2\right) \to (R_3)$ *therefore, we use the matrix*

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{5} & 1 \end{bmatrix}$$

with pre-multiplication. You will notice that if we let $A_3 = E_3 A_2$, *then*

$$A_3 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 0 & -\frac{28}{5} \end{bmatrix}$$

Next, we want to set a_{33} *to* 1. *For this, we want to simulate* $\left(-\frac{5}{28} R_3\right) \to (R_3)$ *therefore, we use the matrix*

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{5}{28} \end{bmatrix}$$

with pre-multiplication. You will notice that if we let $A_4 = E_4 A_3$, *then*

$$A_4 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we want to set a_{22} *to* 1. *For this, we want to simulate* $\left(-\frac{1}{5} R_2\right) \to (R_2)$ *therefore, we use the matrix*

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with pre-multiplication. You will notice that if we let $A_5 = E_5 A_4$, *then*

$$A_5 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{9}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Remark 110 *In the above example, we have created a sequence of matrices* $A, A_1, A_2, A_3, A_4, A_5$ *defined as follows*

$$\begin{aligned} A_1 &= E_1 A \\ A_2 &= E_2 A_1 = E_2 E_1 A \\ A_3 &= E_3 A_2 = E_3 E_2 E_1 A \\ A_4 &= E_4 A_3 = E_4 E_3 E_2 E_1 A \\ A_5 &= E_5 A_4 = E_5 E_4 E_3 E_2 E_1 A \end{aligned}$$

where the matrices $E_1, ..., E_5$ *are elementary matrices. This is what always happens when doing Gaussian elimination. We begin with some matrix* A*. We pre-multiply it with several elementary matrices* $E_1, ..., E_k$ *until the resulting matrix* A_k *where*

$$A_k = E_k E_{k-1} ... E_2 E_1 A$$

is in row-echelon form.

Definition 111 *Let* A *and* B *be* $m \times n$ *matrices. We say that* B *is* ***row-equivalent*** *to* A *if there exists a finite number of elementary matrices* $E_1, ..., E_k$ *such that*

$$B = E_k E_{k-1} ... E_2 E_1 A$$

When we find the inverse of a square matrix A, we transform it into the identity matrix using elementary row transformations. In other words, we find a finite number of elementary matrices $E_1, ..., E_k$ such that

$$I = E_k E_{k-1} ... E_2 E_1 A$$

Therefore, we have the following theorem:

Theorem 112 *A square matrix* A *is invertible if and only if it can be written as the product of elementary matrices.*
Proof. *We need to prove both directions.*

1. *Let us assume that* A *is invertible. Then, as noted above we have* $I = E_k E_{k-1} ... E_2 E_1 A$*. Therefore,*

   $$A = (E_k E_{k-1} ... E_2 E_1)^{-1}$$

   . As noted above, elementary matrices are invertible and since the inverse of a product is the product of the inverses in reverse order, we have

   $$A = E_1^{-1} E_2^{-1} ... E_{k-1}^{-1} E_k^{-1}$$

   Since the inverse of an elementary matrix is itself an elementary matrix, this direction of the result is proven.

2. *Let us assume that* A *is the product of elementary matrices. We know that elementary matrices are invertible. We also know that the product of invertible matrices is also invertible. It follows that* A *is invertible.*

■

Gathering all the information we know about inverses, systems, elementary matrices, we have the following theorem:

Theorem 113 *If A is an $n \times n$ matrix, then the following statements are equivalent:*

1. *A is invertible.*
2. *$A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ column matrix $\mathbf{b}$.*
3. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
4. *A is row equivalent to I_n.*
5. *A can be written as the product of elementary matrices.*

1.5.5 A Method for Inverting Matrices

Given an $n \times n$ matrix A which is nonsingular (we assume it has an inverse), how do we find its inverse? Answering this question amounts to finding an $n \times n$ matrix B satisfying

$$AB = I$$

where I is the identity matrix of the correct size. We introduce the following notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & .. & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & .. & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix}$$

Thus, solving $AB = I$ amounts to finding the entries (b_{ij}). Also, we let B_j denote the j^{th} column of B. In other words,

$$
\begin{aligned}
B_1 &= \begin{bmatrix} b_{11} \\ b_{21} \\ ... \\ b_{n1} \end{bmatrix} \\
B_2 &= \begin{bmatrix} b_{12} \\ b_{22} \\ ... \\ b_{n2} \end{bmatrix} \\
&... \\
B_n &= \begin{bmatrix} b_{1n} \\ b_{2n} \\ ... \\ b_{nn} \end{bmatrix}
\end{aligned}
$$

Similarly, we let I_j denote the j^{th} column of I. So,

$$
\begin{aligned}
I_1 &= \begin{bmatrix} 1 \\ 0 \\ ... \\ 0 \end{bmatrix} \\
I_2 &= \begin{bmatrix} 0 \\ 1 \\ ... \\ 0 \end{bmatrix} \\
&... \\
I_n &= \begin{bmatrix} 0 \\ 0 \\ ... \\ 1 \end{bmatrix}
\end{aligned}
$$

.

Then, we see that solving the equation

$$AB = I$$

is the same as solving

$$
\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & .. & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}
\begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & .. & ... \\ b_{n1} & b_{n2} & ... & b_{nn} \end{bmatrix}
=
\begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & .. & ... \\ 0 & 0 & ... & 1 \end{bmatrix}
$$

This matrix equation is equivalent to solving the n systems

$$AB_j = I_j \text{ for } j = 1, 2, ..., n$$

We know how to solve each of these systems, for example using Gauss-Jordan elimination. Since the transformations involved in Gauss-Jordan elimination only depend on the coefficient matrix A, we will realize that we are repeating a lot of the work if we solve the n systems. It is more efficient to do the following:

1. Form the matrix $[A : I]$ by adjoining A and I. Note it will be an $n \times 2n$ matrix.

2. If possible, row reduce A to I using Gauss-Jordan elimination on the entire matrix $[A : I]$. The result will be the matrix $\left[I : A^{-1}\right]$. If this is not possible, then A does not have an inverse.

We illustrate the procedure by doing some examples.

Example 114 *Find the inverse of* $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$

In other words, we want to find $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ *such that* $AB = I$. *We begin by adjoining the identity matrix to* A *to obtain:*

$$\begin{bmatrix} 2 & -17 & 11 & \vdots & 1 & 0 & 0 \\ -1 & 11 & -7 & \vdots & 0 & 1 & 0 \\ 0 & 3 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

We then row reduce A *to the identity matrix by performing Gauss-Jordan elimination to the whole matrix. The transformation* $\left(E_2 + \frac{1}{2}E_1\right) \to (E_2)$ *produces*

$$\begin{bmatrix} 2 & -17 & 11 & \vdots & 1 & 0 & 0 \\ 0 & \frac{5}{2} & -\frac{3}{2} & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & 3 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

The transformation $(2E_2) \to (E_2)$ *produces*

$$\begin{bmatrix} 2 & -17 & 11 & \vdots & 1 & 0 & 0 \\ 0 & 5 & -3 & \vdots & 1 & 2 & 0 \\ 0 & 3 & -2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

The transformation $\left(E_3 - \frac{3}{5}E_2\right) \to (E_3)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 2 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & -\frac{6}{5} & 1 \end{array}\right]$$

The transformation $(-5E_3) \to (E_3)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 5 & -3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

The transformation $(E_2 + 3E_3) \to (E_2)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 5 & 0 & 10 & 20 & -15 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

The transformation $\left(\frac{1}{5}E_2\right) \to (E_2)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

The transformation $(E_1 - 11E_3) \to (E_1)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & -17 & 0 & -32 & -66 & 55 \\ 0 & 1 & 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

The transformation $(E_1 + 17E_2) \to (E_1)$ *produces*

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

Finally, the transformation $\left(\frac{1}{2}E_1\right) \rightarrow (E_1)$ *produces*

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 4 & -3 \\ 0 & 0 & 1 & 3 & 6 & -5 \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

Example 115 *Find the inverse of* $B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 0 \\ -1 & 3 & -3 \end{bmatrix}$.
We begin by adjoining the identity matrix to B. *We obtain*

$$\left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ -1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

Next, we try to row reduce B *to the identity matrix by applying Gauss-Jordan elimination to the whole matrix. Performing* $(E_2 - 2E_1) \rightarrow (E_2)$ *produces*

$$\left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ -1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

Performing $(E_3 + E_1) \rightarrow (E_3)$ *produces*

$$\left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 1 \end{array}\right]$$

The transformation $(E_3 - E_2) \rightarrow (E_3)$ *produces*

$$\left[\begin{array}{ccc:ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{array}\right]$$

We do not need to continue, the last row of what used to be B *consists entirely of* $0's$. B *does not have an inverse. Another way of saying this is that* B *is singular.*

We finish this section by revisiting a theorem about the cancellation laws.

Theorem 116 *If C is an invertible matrix, then the following is true.*

1. *If $AC = BC$, then $A = B$. This is called the right cancellation property.*
2. *If $CA = CB$, then $A = B$. This is called the left cancellation property.*

Proof. *To prove part 1, we use the fact that C is invertible*

$$\begin{aligned} AC &= BC \Rightarrow (AC)\,C^{-1} = (BC)\,C^{-1} \\ &\Rightarrow A\left(CC^{-1}\right) = B\left(CC^{-1}\right) \\ &\Rightarrow AI = BI \\ &\Rightarrow A = B \end{aligned}$$

Part 2 is proven the same way. ■

Remark 117

1. *We must list the right and left cancellation properties. Because matrix multiplication is not commutative, having one of them does not necessarily imply that the other one is also true.*
2. *When it comes to the cancellation property, matrices behave like real numbers. The cancellation property for real numbers says that $ac = bc \Rightarrow a = b$ if $c \neq 0$. This is the same as saying $ac = bc \Rightarrow a = b$ if c has an inverse, because, for real numbers, only 0 does not have an inverse. That is exactly what the cancellation property for matrices says.*

1.6 Problems

1. Using your knowledge of diagonal matrices and inverse matrices, find a general formula for the inverse of a diagonal matrix. Do all diagonal matrices have an inverse?
2. Decide whether each statement below is True or False. Justify your answer by citing a theorem, or giving a counter example.
   (a) If A is invertible, then the system $A\mathbf{x} = \mathbf{b}$ is consistent.
   (b) If A is not invertible, then the system $A\mathbf{x} = \mathbf{b}$ is consistent.
   (c) If A is not invertible, then the system $A\mathbf{x} = \mathbf{b}$ is not consistent.
   (d) If A is not invertible, then the system $A\mathbf{x} = \mathbf{0}$ is consistent.
   (e) If A is not invertible, then the system $A\mathbf{x} = \mathbf{0}$ is not consistent.
3. On page 57 - 59, do the following problems: 1, 2, 5, 6, 9, 10, 11, 12, 18, 21.