Example for elementary matrices and finding the inverse

1. Let

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Find elementary matrices E_1, E_2 and E_3 such that $E_3E_2E_1A = I_3$.

Solution:

- We can multiply row 2 by $\frac{1}{4}$ in order to get a leading one in the second row. The corresponding elementary matrix for this row operation is $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By multiplying A on the left with E_1, we obtain the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

- Now add 2 times row 3 to row 1 in order to clear the -2 in the first row. The corresponding elementary matrix for this row operation is $E_2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We obtain the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

- The last row operation is to subtract $\frac{3}{4}$ times row 3 from row 2. The corresponding elementary matrix is $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$, to obtain

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus we have E_1, E_2, E_3 such that $E_3E_2E_1A = I_3$. You may check your answer by multiplying the 4 matrices on the left hand side and seeing if you obtain the identity matrix.

Remark: E_1, E_2 and E_3 are not unique. If you used different row operations in order to obtain the RREF of the matrix A, you would get different elementary matrices.

(b) Write A as a product of elementary matrices.

Solution: From part (a), we have that $E_3E_2E_1A = I_3$. Below is one way to see that $A = E_1^{-1}E_2^{-1}E_3^{-1}$. We can multiply the equation by E_3^{-1} on the left on both sides of the equation.

$$
\begin{aligned}
E_3E_2E_1A &= I_3 \\
E_3^{-1}E_3E_2E_1A &= E_3^{-1}I_3 \\
E_2E_1A &= E_3^{-1}.
\end{aligned}
$$

Repeating with E_2 and E_1:

$$
\begin{aligned}
E_2E_1A &= E_3^{-1} \\
E_1A &= E_2^{-1}E_3^{-1} \\
A &= E_1^{-1}E_2^{-1}E_3^{-1}.
\end{aligned}
$$

We discussed how to find the inverse of an elementary matrix in class. E_3^{-1} is the matrix we multiply E_3 with in order to obtain the identity matrix, and it represents the inverse operation. For instance, for E_3, the matrix E_3^{-1} represents the row operation of *adding* $\frac{3}{4}$ times row 3 to row 2.

$$
A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}
$$

(c) If instead of this, we were *only* asked to find A^{-1}, we would do the following:

$$
\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
$$

$$
\begin{matrix} R_1 \\ \frac{1}{4}R_2 \\ R_3 \end{matrix} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
$$

$$
\begin{matrix} R_1 + 2R_3 \\ R_2 \\ R_3 \end{matrix} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
$$

$$
\begin{matrix} R_1 \\ R_2 - \frac{3}{4}R_3 \\ R_3 \end{matrix} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)
$$

Thus,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

You may check by multiplying that

$$AA^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & \frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Suppose that A is a 3×3 matrix, $A^T + 5I$ is invertible, and

$$(A^T + 5I)^{-1} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

Find A.

Answer:

$$A = \begin{pmatrix} -4 & 0 & -2 \\ 3 & -6 & -4 \\ 0 & 0 & -4 \end{pmatrix}$$