Chapter 6

Linear Transformation

6.1 Intro. to Linear Transformation

Homework: [Textbook, §6.1 Ex. 3, 5, 9, 23, 25, 33, 37, 39 ,53, 55, 57, 61(a,b), 63; page 371-].

In this section, *we discuss linear transformations.*

Recall, from calculus courses, a funtion

$$f : X \to Y$$

from a set X to a set Y associates to each $x \in X$ a unique element $f(x) \in Y$. Following is some commonly used terminologies:

1. X is called the domain of f.
2. Y is called the codomain of f.
3. If $f(x) = y$, then we say y is the image of x. **The preimage** of y is
$$preimage(y) = \{x \in X : f(x) = y\}.$$
4. The range of f is the set of images of elements in X.

In this section we deal with functions from a vector sapce V to another vector space W, that respect the vector space structures. Such a function will be called a linear transformation, defined as follows.

Definition 6.1.1 Let V and W be two vector spaces. A function

$$T : V \to W$$

is called a **linear transformation** of V into W, if following two prperties are true for all $\mathbf{u}, \mathbf{v} \in V$ and scalars c.

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u})+T(\mathbf{v})$. (*We say that T preserves additivity.*)
2. $T(c\mathbf{u}) = cT(\mathbf{u})$. (*We say that T preserves scalar multiplication.*)

Reading assignment Read [Textbook, Examples 1-3, p. 362-].

Trivial Examples: Following are two easy exampes.

1. Let V, W be two vector spaces. Define $T : V \to W$ as

$$T(\mathbf{v}) = \mathbf{0} \quad for \quad all \quad \mathbf{v} \in V.$$

Then T is a linear transformation, to be called the **zero transformation.**

2. Let V be a vector space. Define $T : V \to V$ as

$$T(\mathbf{v}) = \mathbf{v} \quad for \quad all \quad \mathbf{v} \in V.$$

Then T is a linear transformation, to be called the **identity transformation of** V.

6.1.1 Properties of linear transformations

Theorem 6.1.2 Let V and W be two vector spaces. Suppose $T : V \to W$ is a linear transformation. Then

1. $T(\mathbf{0}) = \mathbf{0}$.

2. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v} \in V$.

3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$.

4. If

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}$$

then

$$T(\mathbf{v}) = T\left(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}\right) = c_1T\left(\mathbf{v_1}\right) + c_2T\left(\mathbf{v_2}\right) + \cdots + c_nT\left(\mathbf{v_n}\right).$$

Proof. By property (2) of the definition 6.1.1, we have

$$T(\mathbf{0}) = T(0\mathbf{0}) = 0T(\mathbf{0}) = \mathbf{0}.$$

So, (1) is proved. Similarly,

$$T(-\mathbf{v}) = T((-1)\mathbf{v}) = (-1)T(\mathbf{v}) = -T(\mathbf{v}).$$

So, (2) is proved. Then, by property (1) of the definition 6.1.1, we have

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u} + (-1)\mathbf{v}) = T(\mathbf{u}) + T((-1)\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}).$$

The last equality follows from (2). So, (3) is proved.

To prove (4), we use induction, on n. For $n = 1$: we have $T(c_1\mathbf{v_1}) = c_1T(\mathbf{v_1})$, by property (2) of the definition 6.1.1.

For $n = 2$, by the two properties of definition 6.1.1, we have

$$T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = T(c_1\mathbf{v_1}) + T(c_2\mathbf{v_2}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}).$$

So, (4) is prove for $n = 2$. Now, we assume that the formula (4) is valid for $n - 1$ vectors and prove it for n. We have

$$T\left(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_n\mathbf{v_n}\right) = T\left(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_{n-1}\mathbf{v_{n-1}}\right) + T\left(c_n\mathbf{v_n}\right)$$

$$= \left(c_1T\left(\mathbf{v_1}\right) + c_2T\left(\mathbf{v_2}\right) + \cdots + c_{n-1}T\left(\mathbf{v_{n-1}}\right)\right) + c_nT(\mathbf{v_n}).$$

So, the proof is complete. ∎

6.1.2 Linear transformations given by matrices

Theorem 6.1.3 Suppose A is a matrix of size $m \times n$. Given a vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \quad define \quad T(\mathbf{v}) = A\mathbf{v} = A \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}.$$

Then T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof. From properties of matrix multiplication, for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalar c we have

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT(\mathbf{u}).$$

The proof is complete. ■

Remark. Most (or all) of our examples of linear transformations come from matrices, as in this theorem.

Reading assignment Read [Textbook, Examples 2-10, p. 365-].

6.1.3 Projections along a vector in $\mathbb{R}^n$

Projections in $\mathbb{R}^n$ is a good class of examples of linear transformations. We define projection along a vector.

Recall the definition 5.2.6 of orthogonal projection, in the context of Euclidean spaces $\mathbb{R}^n$.

Definition 6.1.4 Suppose $\mathbf{v} \in \mathbb{R}^n$ is a vector. Then,

$$for \quad \mathbf{u} \in \mathbb{R}^n \quad define \qquad proj_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v} \cdot \mathbf{u}}{\| \mathbf{v} \|^2}\mathbf{v}$$

1. Then $proj_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation.

   Proof. This is because, for another vector $\mathbf{w} \in \mathbb{R}^n$ and a scalar c, it is easy to check

   $$proj_{\mathbf{v}}(\mathbf{u}+\mathbf{w}) = proj_{\mathbf{v}}(\mathbf{u}) + proj_{\mathbf{v}}(\mathbf{w}) \quad and \quad proj_{\mathbf{v}}(c\mathbf{u}) = c\,(proj_{\mathbf{v}}(\mathbf{u}))\,.$$

2. The point of such projections is that any vector $\mathbf{u} \in \mathbb{R}^n$ can be written uniquely as a sum of a vector along $\mathbf{v}$ and another one perpendicular to $\mathbf{v}$:

   $$\mathbf{u} = proj_{\mathbf{v}}(\mathbf{u}) + (\mathbf{u} - proj_{\mathbf{v}}(\mathbf{u}))\,.$$

It is easy to check that $(\mathbf{u} - proj_{\mathbf{v}}(\mathbf{u})) \perp proj_{\mathbf{v}}(\mathbf{u})$.

Exercise 6.1.5 (Ex. 4, p. 371) Let

$$T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3)$$

1. Compute $T(-4, 5, 1)$.

   Solution:

   $T(-4, 5, 1) = (2*(-4)+5, 2*5-3*(-4), -4-1) = (-3, 22, -5).$

2. Compute the preimage of $\mathbf{w} = (4, 1, -1)$.

   Solution: Suppose (v_1, v_2, v_3) is in the preimage of $(4, 1, -1)$. Then

   $$(2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3) = (4, 1, -1).$$

   So,

   $$\begin{array}{cccc} 2v_1 & +v_2 & & = 4 \\ & 2v_2 & -3v_3 & = 1 \\ v_1 & & -v_3 & = -1 \end{array}$$

   The augmented matrix of this system is

   $$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & 2 & -3 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \quad its\, Gauss\!-\!Jordan\, form \quad \begin{bmatrix} 1 & 0 & 0 & .5714 \\ 0 & 1 & 0 & 2.85714 \\ 0 & 0 & 1 & 1.5714 \end{bmatrix}$$

   So,

   $$Preimage((4, 1, -1)) = \{(.5714, 2.85714, 1.5714)\}.$$

Exercise 6.1.6 (Ex. 10. p. 371) Determine whether the function

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x, y) = (x^2, y) \quad is \quad linear?$$

Solution: We have

$$T((x, y) + (z, w)) = T(x + z, y + w) = ((x + z)^2, y + w)$$
$$\neq (x^2, y) + (z^2, w) = T(x, y) + T(z, w).$$

So, T does not preserve additivity. So, T is not linear.

Alternately, you could check that T does not preserve scalar multiplication.

Alternately, you could check this failure(s), numerically. For example,

$$T((1, 1) + (2, 0)) = T(3, 1) = (9, 1) \neq T(1, 1) + T(2, 0).$$

Exercise 6.1.7 (Ex. 24, p. 371) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1, 0, 0) = (2, 4, -1), \quad T(0, 1, 0) = (1, 3, -2), \quad T(0, 0, 1) = (0, -2, 2).$$

Compute $T(-2, 4, -1)$.

Solution: We have

$$(-2, 4, -1) = -2(1, 0, 0) + 4(0, 1, 0) - (0, 0, 1).$$

So, $T(-2, 4, -1) =$

$$-2T(1, 0, 0)+4T(0, 1, 0)-T(0, 0, 1) = (2, 4, -1)+(1, 3, -2)+(0, -2, 2) = (3, 5, -1).$$

Remark. A linear transformation $T : V \to V$ can be defined, simply by assigning values $T(\mathbf{v_i})$ for any basis $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ of V. In this case of the our problem, values were assigned for the standard basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ of $\mathbb{R}^3$.

Exercise 6.1.8 (Ex. 38, p. 372) Let

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}. \quad \textit{Let} \quad T : \mathbb{R}^5 \to \mathbb{R}^2$$

be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$.

1. Compute $T(1, 0, -1, 3, 0)$.

   Solution:

$$T(1, 0, -1, 3, 0) = \begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}.$$

2. Compute preimage, under T, of $(-1, 8)$.

   Solution: The preimage consists of the solutions of the linear system

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

   The augmented matrix of this system is

$$\begin{bmatrix} -1 & 2 & 1 & 3 & 4 & -1 \\ 0 & 0 & 2 & -1 & 0 & 8 \end{bmatrix}.$$

   The Gauss-Jordan form is

$$\begin{bmatrix} 1 & -2 & 0 & -3.5 & -4 & 5 \\ 0 & 0 & 1 & -.5 & 0 & 4 \end{bmatrix}.$$

We use parameters $x_2 = t, x_4 = s, x_5 = u$ and the solotions are given by

$$x_1 = 5 + 2t + 3.5s + 4u, x_2 = t, x_3 = 4 + .5s, x_4 = s, x_5 = u$$

So, the preimage

$$T^{-1}(-1, 8) = \{(5+2t+3.5s+4u, \ \ t, \ \ 4+.5s, \ \ s, \ \ u) : t, s, u \in \mathbb{R}\}.$$

Exercise 6.1.9 (Ex. 54 (edited), p. 372) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(1, 1) = (0, 2)$ and $T(1, -1) = (2, 0)$.

1. Compute $T(1, 4)$.

   Solution: We have to write

   $$(1, 4) = a(1, 1)+b(1, -1). \quad \textit{Solving} \quad (1, 4) = 2.5(1, 1)-1.5(1, -1).$$

   So,

   $$T(1, 4) = 2.5T(1, 1)-1.5T(1, -1) = 2.5(0, 2)-1.5(2, 0) = (-3, 5).$$

2. Compute $T(-2, 1)$.

   Solution: We have to write

   $$(-2, 1) = a(1, 1)+b(1, -1). \quad \textit{Solving} \quad (-2, 1) = -.5(1, 1)-1.5(1, -1).$$

   So,

   $$T(-2, 1) = -.5T(1, 1)-1.5T(1, -1) = -.5(0, 2)-1.5(2, 0) = (-3, -1).$$

Exercise 6.1.10 (Ex. 61 (edited), p. 372) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ the projection $T(\mathbf{u}) = proj_{\mathbf{v}}(\mathbf{u})$ where $\mathbf{v} = (1, 1, 1)$.

1. Find $T(x, y, z)$.

   Solution: See definition 6.1.4.

$$proj_{\mathbf{v}}(x, y, z) = \frac{\mathbf{v} \cdot (x, y, z)}{\| \mathbf{v} \|^2}\mathbf{v} =$$

$$\frac{(1,1,1) \cdot (x, y, z)}{\| (1,1,1) \|^2}(1,1,1) = \frac{x+y+z}{3}(1,1,1)$$
$$= \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right).$$

2. Compute $T(5, 0, 5)$.

   Solution: We have

$$T(5,0,5) = \left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right) = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right).$$

3. Compute the matrix of T.

   Solution: The matrix is given by

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

   because

$$T(x, y, z) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

6.2 Kernel and Range of linear Transformation

We will essentially, skip this section. Before we do that, let us give a few definitions.

Definition 6.2.1 Let V, W be two vector spaces and $T : V \to W$ a linear transformation.

1. Then the kernel of T, denoted by $ker(T)$, is the set of $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$. Notationally,

$$ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}.$$

   It is easy to see the $ker(T)$ is a subspace of V.

2. Recall, range of T, denoted by $range(T)$, is given by

$$range(T) = \{\mathbf{v} \in W : \mathbf{w} = T(\mathbf{v}) \quad for \quad some \quad \mathbf{v} \in V\}.$$

   It is easy to see the $range(T)$ is a subspace of W.

3. We say the T is **isomorphism**, if T is one-to-one and onto. *It follows, that T is an isomorphism if $ker(T) = \{\mathbf{0}\}$ and $range(T) = W$.*

6.3 Matrices for Linear Transformations

Homework: *[Textbook, §6.3, Ex. 5, 7, 11, 13, 17, 19, 21, 23, 25, 29, 31, 33, 35(a,b), 37(a,b), 39, 43, 45, 47; p. 397]*

Optional Homework: *[Textbook, §6.3, Ex. 57, 59; p. 398]. (We will not grade them.)* **In this section,** *to each linear transformation we*

associate a matrix.

Linear transformations and matrices have very deep relationships. In fact, study of linear transformations can be reduced to the study of matrices and conversely. *First, we will study this relationship for linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$; and later study the same for linear transformations $T : V \to W$ of general vector spaces.*

In this section, we will denote the vectors in $\mathbb{R}^n$, as column matrices. Recall, written as columns, the standard basis of $\mathbb{R}^n$ is given by

$$B = \{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}\} \left\{ \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{bmatrix}. \right\}$$

Theorem 6.3.1 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Write

$$T(\mathbf{e_1}) = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, T(\mathbf{e_2}) = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix}, \cdots, T(\mathbf{e_n}) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}.$$

These columns $T(\mathbf{e_1}), T(\mathbf{e_2})\ldots, T(\mathbf{e_n})$ form a $m\times n$ matrix A as follows,

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

This matrix A has the property that

$$T(\mathbf{v}) = A\mathbf{v} \qquad for \quad all \quad \mathbf{v} \in \mathbb{R}^n.$$

This matrix A is called the **standard matrix** of T.

Proof. We can write $\mathbf{v} \in \mathbb{R}^n$ as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \ldots \\ v_n \end{bmatrix} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_n\mathbf{e_n}.$$

We have,

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \ldots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{bmatrix}$$

$$= v_1T(\mathbf{e_1}) + v_2T(\mathbf{e_2}) + \cdots + v_nT(\mathbf{e_n}) = T(v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_n\mathbf{e_n}) = T(\mathbf{v}).$$

The proof is complete. ■

Reading assignment: Read [Textbook, Examples 1,2; page 389-390].

In our context of linear transformations, we recall the following definition of composition of functions.

Definition 6.3.2 Let

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m, \quad T_2 : \mathbb{R}^m \to \mathbb{R}^p$$

be two linear transformations. Define the **composition** $T : \mathbb{R}^n \to \mathbb{R}^p$ of T_1 with T_2 as

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) \quad for \quad \mathbf{v} \in \mathbb{R}^n.$$

The composition T is denoted by $T = T_2 o T_1$. Diagramatically,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T_1"] \arrow[dr, "T=T_2oT_1"'] & \mathbb{R}^m \arrow[d, "T_2"] \\ & \mathbb{R}^p. \end{tikzcd}$$

Theorem 6.3.3 Suppose

$$T_1 : \mathbb{R}^n \to \mathbb{R}^m, \quad T_2 : \mathbb{R}^m \to \mathbb{R}^p$$

are two linear transformations.

1. Then, the composition $T = T_2 o T_1 : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation.

2. Suppose A_1 is the standard matrix of T_1 and A_2 is the standard matrix of T_2. Then, the standard matrix of the composition

$$T = T_2 o T_1 \quad is \quad the \quad product \quad A = A_2 A_1.$$

Proof. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and scalars c, we have

$$T(\mathbf{u}+\mathbf{v}) = T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u})+T_1\mathbf{v})) = T_2(T_1(\mathbf{u}))+T_2T_1(\mathbf{v})) = T(\mathbf{u})+T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = T_2(T_1(c\mathbf{u})) = T_2(cT_1(\mathbf{u})) = cT_2(T_1(\mathbf{u})) = cT(\mathbf{u}).$$

So, T preserves addition and scalar multiplication. Therefore T is a linear transformation and (1) is proved. To prove (2), we have

$$T(\mathbf{u}) = T_2(T_1(\mathbf{u})) = T_2(A_1\mathbf{u}) = A_2(A_1\mathbf{u}) = (A_2A_1)\mathbf{u}.$$

Therefore $T(\mathbf{e_1})$ is the first column of A_2A_1, $T(\mathbf{e_2})$ is the second column of A_2A_1, and so on. Therefore, the standard matrix of T is A_2A_1. The proof is complete. ■

Reading assignment: Read [Textbook, Examples 3; page 392].

Definition 6.3.4 Let

$$T_1 : \mathbb{R}^n \to \mathbb{R}^n, \quad T_2 : \mathbb{R}^n \to \mathbb{R}^n$$

be two linear transformations such that for every $\mathbf{v} \in \mathbb{R}^n$ we have

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \textit{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v},$$

then we say that T_2 is the **inverse** of T_1, and we say that T_1 is **invertible.**

Such an inverse T_2 of T_1 is unique and is denoted by T_1^{-1}.

*(***Remark.*** Let End$(\mathbb{R}^n)$ denote the set of all linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$. Then End$(\mathbb{R}^n)$ has a binary operation by composition. The identity operation $I : \mathbb{R}^n \to \mathbb{R}^n$ acts as the identitiy under this composition operation. The definition of inverse of T_1 above, just corresponds to the inverse under this composition operation.)*

Theorem 6.3.5 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformations and let A be the standard matrix of T. Then, the following are equivalent,

1. T is invertible.

2. T is an isomorphism.

3. A is invertible.

And, if T is invertible, then the standard matrix of T^{-1} is A^{-1}.

Proof. (*First, recall definition 6.2.1, that T is isomorphism if T is 1 to 1 and onto.*)

(**Proof of** (1) $\Rightarrow$ (2) :) Suppose T is invertible and T_2 be the inverse. Suppose $T(\mathbf{u}) = T(\mathbf{v})$. Then,

$$\mathbf{u} = T_2(T(\mathbf{u})) = T_2(T(\mathbf{v})) = \mathbf{v}.$$

So, T is 1 to 1. Also, given $\mathbf{u} \in \mathbb{R}^n$ we have

$$\mathbf{u} = T(T_2(\mathbf{u})). \quad So, \quad T \quad onto \quad \mathbb{R}^n.$$

So, T is an isomorphism.

(**Proof of** (2) $\Rightarrow$ (3) :) So, assume T is an isomorphism. Then,

$$A\mathbf{x} = \mathbf{0} \Rightarrow T(\mathbf{x}) = T(\mathbf{0}) \Rightarrow \mathbf{x} = \mathbf{0}.$$

So, $A\mathbf{x} = \mathbf{0}$ has an unique solution. Therefore A is invertible and (3) follows from (2).

(**Proof of** (3) $\Rightarrow$ (1) :) Suppose A is invertible. Let $T_2(\mathbf{x}) = A^{-1}\mathbf{x}$, then T_2 is a linear transformation and it is easily checked that T_2 is the inverse of T. So, (1) follows from (3). The proof is complete. ■

Reading assignment: Read [Textbook, Examples 4; page 393].

6.3.1 Nonstandard bases and general vector spaces

The above discussion about (standard) matrices of linear transformations T had to be restricted to linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$. This wa sbecause $\mathbb{R}^n$ has a standard basis $\{\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n},\}$ that we could use.

Suppose $T : V \to W$ is a linear transformation between two abstract vector spaces V, W. Since V and W has no standard bases, we cannot associate a matrix to T. But, if we fix a basis B of V and B' of W we can associate a matrix to T. We do it as follows.

Theorem 6.3.6 Suppose $T : V \to W$ is a linear transformation between two vector spaces V, W. Let

$$B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\} \quad be \quad basis \quad of \quad V$$

and

$$B' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_m}\} \quad be \quad basis \quad of \quad w.$$

We can write

$$T(\mathbf{v_1}) = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \cdots & \mathbf{w_m} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}$$

Writing similar equations for $T(\mathbf{v_2}), \ldots, T(\mathbf{v_n})$, we get

$$\begin{bmatrix} T(\mathbf{v_1}) & T(\mathbf{v_2}) & \cdots & T(\mathbf{v_n}) \end{bmatrix} = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \cdots & \mathbf{w_m} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Then, for $\mathbf{v} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n}$, We have

$$T(\mathbf{v}) = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \cdots & \mathbf{w_m} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}.$$

Write $A = [a_{ij}]$. Then, T is determined by A, with respect to bases B, B'.

Exercise 6.3.7 (Ex. 6, p. 397) Let

$$T(x, y, z) = (5x - 3y + z, 2z + 4y, 5x + 3y).$$

What is the standard matrix of T?

Solution: We have $T : \mathbb{R}^3 \to \mathbb{R}^3$. We write vectors $\mathbf{x} \in \mathbb{R}^3$ as columns

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad instead \qquad of \quad (x, y, z).$$

Recall the standard basis

$$\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad of \quad \mathbb{R}^3.$$

We have

$$T(\mathbf{e_1}) = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} -3 \\ 4 \\ 3 \end{bmatrix}, \quad T(\mathbf{e_3}) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

So, the standard matrix of T is

$$A = \begin{bmatrix} 5 & -3 & 1 \\ 0 & 4 & 2 \\ 5 & 3 & 0 \end{bmatrix}.$$

Exercise 6.3.8 (Ex. 12, p. 397) Let

$$T(x, y, z) = (2x + y, 3y - z).$$

Write down the standard matrix of T and use it to find $T(0, 1, -1)$.

Solution: In this case, $T : \mathbb{R}^3 \to \mathbb{R}^2$. With standard basis $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ as in exercise 6.3.7, we have

$$T(\mathbf{e_1}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad T(\mathbf{e_3}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

So, the standard matrix of T is

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix}.$$

Therefore,

$$T(0,1,-1) = A \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

We will write our answer in as a row: $T(0,1,-1) = (1,4)$.

Exercise 6.3.9 (Ex. 18, p. 397) Let T be the reflection in the line $y = x$ in $\mathbb{R}^2$. So, $T(x,y) = (y,x)$.

1. Write down the standard matrix of T.

   Solution: In this case, $T : \mathbb{R}^2 \to \mathbb{R}^2$. With standard basis $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$, we have

   $$T(\mathbf{e_1}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   So, the standard matrix of T is

   $$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

2. Use the standard matrix to compute $T(3,4)$.

   Solution: Of course, we know $T(3,4) = (4,3)$. They want use to use the standard matrix to get the same answer. We have

   $$T(3,4) = A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \textit{or} \quad T(3,4) = (4,3).$$

Lemma 6.3.10 *Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation by a fixed angle θ. Then*

$$T(x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Proof. We can write

$$x = r\cos\phi, \quad y = r\sin\phi, \quad where \quad r = \sqrt{x^2+y^2}, \quad \tan\phi = y/x.$$

By definition

$$T(x,y) = (r\cos(\phi+\theta), r\sin(\phi+\theta)).$$

Using trig-formulas

$$r\cos(\phi+\theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta = x\cos\theta - y\sin\theta$$

and

$$r\sin(\phi+\theta) = r\sin\phi\cos\theta + r\cos\phi\sin\theta = y\cos\theta + x\sin\theta.$$

So,

$$T(x,y) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proof is complete. ■

Exercise 6.3.11 (Ex. 22, p. 397) Let T be the counterclockwise rotation in $\mathbb{R}^2$ by angle 120^o.

1. Write down the standard matrix of T.

   Solution: We use lemma 6.3.10, with $\theta = 120^0$. So, the standard matrix of T is

   $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 120 & -\sin 120 \\ \sin 120 & \cos 120 \end{bmatrix} = \begin{bmatrix} -.5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -.5 \end{bmatrix}.$$

2. Compute $T(2,2)$.

   Solution: We have

$$T(2,2) = A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -.5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -.5 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1-\sqrt{3} \\ -1+\sqrt{3} \end{bmatrix}.$$

   We write our answer as rows: $T(2,2) = \left(-1-\sqrt{3}, -1+\sqrt{3}\right)$.

Exercise 6.3.12 (Ex. 32, p. 397) Let T be the projection on to the vector $\mathbf{w} = (-1,5)$ in $\mathbb{R}^2 : T(\mathbf{u}) = proj_{\mathbf{w}}(\mathbf{u})$.

1. Find the standard matrix.

   Solution: See definition 6.1.4.

$$T(x,y) = proj_{\mathbf{w}}(x,y) = \frac{\mathbf{w}\cdot(x,y)}{\parallel \mathbf{w} \parallel^2}\mathbf{w} = \frac{(1,-5)\cdot(x,y)}{\parallel (1,-5) \parallel^2}(1,-5) = \frac{x-5y}{26}(1,-5)$$
$$= \left(\frac{x-5y}{26}, \frac{-5x+25y}{26}\right).$$

   So, with $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have (write/think everything as columns):

$$T(\mathbf{e_1}) = \begin{bmatrix} \frac{1}{26} \\ -\frac{5}{26} \end{bmatrix}, \qquad T(\mathbf{e_2}) = \begin{bmatrix} -\frac{5}{26} \\ \frac{25}{26} \end{bmatrix}$$

   So, the standard matrix is

$$A = \begin{bmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{bmatrix}.$$

2. Compute $T(2,3)$.

   Solution: We have

$$T(2,3) = A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -.5 \\ \frac{65}{26} \end{bmatrix}.$$

   We write our answer in row-form: $T(2,3) = \left(-.5, \frac{65}{26}\right)$.

Exercise 6.3.13 (Ex. 36, p. 397) Let

$$T(x, y, z) = (3x - 2y + z, 2x - 3y, y - 4z).$$

1. Write down the standard matrix of T.

   Solution: with $\mathbf{e_1} = (1, 0, 0)^T, \mathbf{e_2} = (0, 1, 0)^T, \mathbf{e_3} = (0, 0, 1)^T$ we have (write/think everything as columns):

$$T(\mathbf{e_1}) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}, \quad T(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}.$$

   So, the standard matrix is

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix}.$$

2. Compute $T(2, -1, -1)$.

   Solution: We have

$$T(2, -1, -1) = A \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 3 \end{bmatrix}.$$

Exercise 6.3.14 (Ex. 40, p. 397) Let

$$T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_1(x, y) = (x - 2y, 2x + 3y)$$

and

$$T_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_2(x, y) = (y, 0).$$

Compute the standard matrices of $T = T_2 o T_1$ and $T' = T_1 T_2$.

Solution: We solve it in three steps:

Step-1: First, compute the standard matrix of T_1. With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have (write/think everything as columns):

$$T_1(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T_1(\mathbf{e_2}) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

So, the standard matrix of T_1 is

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}.$$

Step-2: Now, compute the standard matrix of T_2. With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have :

$$T_2(\mathbf{e_1}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad T_2(\mathbf{e_2}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So, the standard matrix of T_2 is

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Step-3 By theorem 6.3.3, the standard matrix of $T = T_2T_1$ is

$$A_2A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}. \quad So, \quad T(x,y) = (2x+3y, 0).$$

Similarly, the standard matrix of $T' = T_1T_2$ is

$$A_1A_2 = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}. \quad So, \quad T'(x,y) = (y, 2y).$$

Exercise 6.3.15 (Ex. 46, p. 397) Determine whether

$$T(x,y) = (x+2y, x-2y).$$

is invertible or not.

Solution: Because of theorem 6.3.5, we will check whether the standard matrix of T is invertible or not.

With $\mathbf{e_1} = (1,0)^T, \mathbf{e_2} = (0,1)^T$ we have :

$$T(\mathbf{e_1}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad T(\mathbf{e_2}) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

So, the standard matrix of T is

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}.$$

Note, $\det A = -4 \neq 0$. So, T is invertible and hence T is invertible.

Exercise 6.3.16 (Ex. 58, p. 397) Determine whether

$$T(x,y) = (x-y, 0, x+y).$$

Use $B = \{\mathbf{v_1} = (1,2), \mathbf{v_2} = (1,1)\}$ as basis of the domain $\mathbb{R}^2$ and $B' = \{\mathbf{w_1} = (1,1,1), \mathbf{w_2} = (1,1,0), \mathbf{w_3} = (0,1,1)\}$ as basis of codomain $\mathbb{R}^3$. Compute matrix of T with respect to B, B'.

Solution: We use theorem 6.3.6. We have

$$T(\mathbf{u_1}) = T(1,2) = (-1,0,3), \quad T(\mathbf{u_2}) = T(1,1) = (0,0,2).$$

We solve the equation:

$$(-1,0,3) = a\mathbf{w_1} + b\mathbf{w_2} + c\mathbf{w_3} = a(1,1,1) + b(1,1,0) + c(0,1,1)$$

and we have

$$(-1,0,3) = 2(1,1,1) - 3(1,1,0) + 1(0,1,1) = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}.$$

Similarly, we solve

$$(0,0,2) = a\mathbf{w_1} + b\mathbf{w_2} + c\mathbf{w_3} = a(1,1,1) + b(1,1,0) + c(0,1,1)$$

and we have

$$(0,0,2) = 2(1,1,1) - 2(1,1,0) + 0(0,1,1) = \begin{bmatrix} \mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.$$

So, the matrix of T with respect to the bases B, B' is

$$A = \begin{bmatrix} 2 & 2 \\ -3 & -2 \\ 1 & 0 \end{bmatrix}.$$

6.4 Transition Matrices and Similarity

We will skip this section. I will just explain the section heading. You know what are Transition matrices of a linear transformation $T : V \to W$. They are a matrices described in theorem 6.3.6.

Definition 6.4.1 Suppose A, B are two square matrices of size $n \times n$. We say A, B are **similar,** if $A = P^{-1}BP$ for some invertible matrix P.

6.5 Applications of Linear Trans.

Homework: [Textbook, §6.5 Ex. 11 (a), 13 (a), 25, 27, 29, 35, 37, 39, 43, 49, 51, 53, 55, 63, 65; page 414-415].

In this section, *we discuss geometric interpretations of linear transformations represented by* 2×2 *elementary matrices.*

Proposition 6.5.1 Let A be a 2×2 matrix and

$$T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix}.$$

We will write the right hand side as a row, *which is an abuse of natation.*

1. If

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad then \quad T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (-x, y)^T$$

   represents the reflection in $y-$axis. See [Textbook, Example 1 (a), p.407] for the diagram.

2. If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad then \quad T(x, y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (x, -y)^T$$

   represents the reflection in $x-$axis. See [Textbook, Example 1 (b), p.407] for the diagram.

3. If
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad then \quad T(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (y,x)^T$$
represents the reflection in line $y = x$. See [Textbook, Example 1 (c), p.407] for the diagram.

4. If
$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \quad then \quad T(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (kx,y)^T.$$
If $k > 1$, then T represents expansion in horizontal direction and $0 < k < 1$, then T represents contraction in horizontal direction. See [Textbook, Example 2 Fig 6.12 , p.409] for diagrams.

5. If
$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \quad then \quad T(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (x,ky)^T.$$
If $k > 1$, then T represents expansion in vertical direction and $0 < k < 1$, then T represents contraction in vertical direction. See [Textbook, Example 2 Fig 6.13, p.409] for diagrams.

6. If
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad then \quad T(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (x+ky,y)^T.$$
Then T represents horizontal shear. (Assume $k > 0$.) The upper-half plane are sheared to right and lower-half plane are sheared to left. The points on the $x-$axis reamain fixed. See [Textbook, Example 3, fig 6.14, p.409] for diagrams.

7. If
$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad then \quad T(x,y) = A \begin{bmatrix} x \\ y \end{bmatrix} = (x, kx+y)^T.$$
Then T represents vertical shear. (Assume $k > 0$.) The right-half-plane are sheared to upward and left-half-plane are sheared to downward. The points on the $y-$axis reamain fixed. See [Textbook, Example 3, fig 6.15, p.409] for diagrams.

6.5.1 Computer Graphics

Linear transformations are used in computer graphics to move figures on the computer screens. I am sure all kinds of linear (and nonlinear) transformations are used. Here, we will only deal with rotations by an angle θ, around (1) $x-$axis, (2) $y-$axis and (3) $z-$axis as follows:

Proposition 6.5.2 Suppose θ is an angle. Suppose we want to rotate the point (x, y, z) counterclockwise about $z-$axis through an angle θ. Let us denote this transformation by T and write $T(x, y, z) = (x', y', z')^T$. Then using lemma 6.3.10, we have

$$T(x,y,z) = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}.$$

Similarly, we can write down the linear transformations corresponding to rotation around $x-$axis and $y-$axis. We write down the transition matrices for these three matrices as follows:

1. The standard matrix for this transformation of counterclockwise

rotation by an angle θ, about $x-$axis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

2. The standard matrix for this transformation of counterclockwise rotation by an angle θ, about $y-$axis is

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

3. The standard matrix for this transformation of counterclockwise rotation by an angle θ, about $z-$axis is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

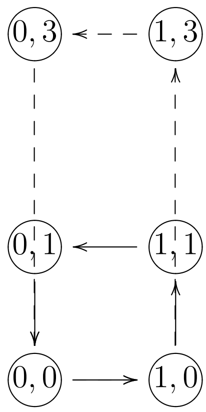
Reading assignment: Read [Textbook, Examples 4 and 5; page 410-412].

Exercise 6.5.3 (Ex. 26, p. 414) Let $T(x, y) = (x, 3y)$ (This is a vertical expansion.) Sketch the image of the unit square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$.

Solution: We have

$$T(0, 0) = (0, 0), \quad T(1, 0) = (1, 0), \quad T(1, 1) = (1, 3), \quad T(0, 1) = (0, 3).$$

Diagram:



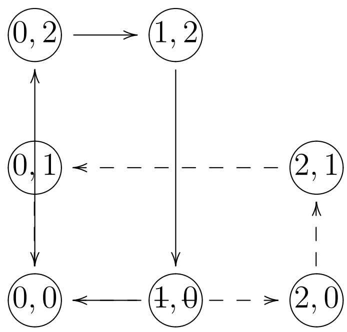
Here, the solid arrows represent the original rectangle and the brocken arrows represent the image.

Exercise 6.5.4 (Ex. 30, p. 414) Let T be the reflection in the line $y = x$. Sketch the image of the rectangle with vertices $(0,0), (0,2), (1,2), (1,0)$.

Solution: Recall, (see Proposition 6.5.1 (3)) that $T(x, y) = (y, x)$. We have

$$T(0,0) = (0,0), T(0,2) = (2,0), T(1,2) = (2,1), T(1,0) = (0,1).$$

Diagram:



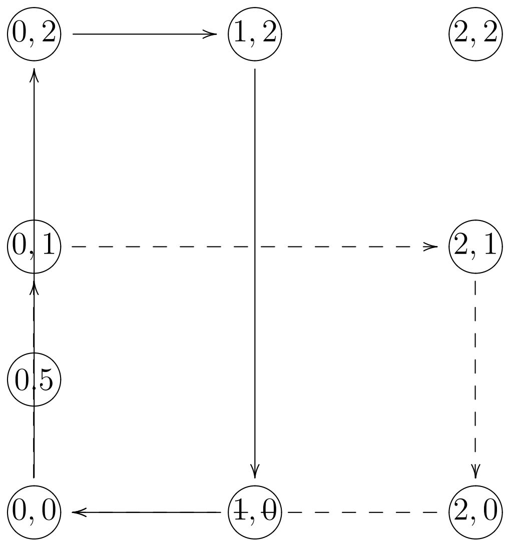
Here, the solid arrows represent the original rectangle and the brocken arrows represent the image.

Exercise 6.5.5 (Ex. 38, p. 414) Suppose T is the expansion and contraction represented by $T(x, y) = \left(2x, \frac{y}{2}\right)$. Sketch the image of the rectangle with vertices $(0, 0), (0, 2), (1, 2), (1, 0)$.

Solution: Recall, (see Proposition 6.5.1 (3)) that $(x, y) = (y, x)$.We have

$$T(0, 0) = (0, 0), T(0, 2) = (0, 1), T(1, 2) = (2, 1), T(1, 0) = (2, 0).$$

Diagram:



Here, the solid arrows represent the original rectangle and thebrocken arrows represent the image.

Exercise 6.5.6 (Ex. 44, p. 414) Give the geometric description of the linear transformation defined by the elementary matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Solution: By proposition 6.5.1 (6) this is a horizontal shear. Here, $T(x, y) = (x + 3y, y)$.

Exercise 6.5.7 (Ex. 50 and 54, p. 415) Find the matrix of the transformation T that will produce a 60^o rotation about the $x-$axis. Then compute the image $T(1,1,1)$.

Solution: By proposition 6.5.2 (1) the matrix is given by $A =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60^o & -\sin 60^o \\ 0 & \sin 60^o & \cos 60^o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

So,

$$T(1,1,1) = A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{bmatrix}.$$

Exercise 6.5.8 (Ex. 64, p. 415) Determine the matrix that will produce a 45^o rotation about the $y-$axis followed by 90^o rotation about the $z-$axis. Then also compute the image of the line segment from $(0,0,0)$ to $(1,1,1)$.

Solution: We will do it in three (or four) steps.

Step-1 Let T_1 be the rotation by 45^o about the $y-$axis. By proposition 6.5.2 (2) the matrix of T_1 is given by $A =$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos 45^0 & 0 & \sin 45^0 \\ 0 & 1 & 0 \\ -\sin 45^0 & 0 & \cos 45^0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Step-2 Let T_2 be the rotation by 90^o rotation about the $z-$axis. By proposition 6.5.2 (3) the matrix of T_2 is given by $B =$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 90^o & -\sin 90^o & 0 \\ \sin 90^o & \cos 90^o & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step-3 So, thematrix of the composite transformation $T = T_2T_1$ is matrix

$$BA = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The Last Part: So,

$$T(1,1,1) = BA \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ \sqrt{2} \\ 0 \end{bmatrix}.$$