18 Orthogonality and related matters

18.1 Orthogonality

Recall that two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal if $\mathbf{x}\bullet\mathbf{y} = 0$. (This is the Greek version of "perpendicular".)

Example: The two vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

are orthogonal, since their dot product is $(2)(1) + (2)(-1) + (4)(0) = 0$.

☐ Definition: A set of non-zero vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be **mutually orthogonal** if $\mathbf{v}_i\bullet\mathbf{v}_j = 0$ for all $i \neq j$.

The standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ are mutually orthogonal.

The vector $\mathbf{0}$ is orthogonal to everything.

☐ Definition: A **unit vector** is a vector of length 1. If its length is 1, then the square of its length is also 1. So $\mathbf{v}$ is a unit vector $\iff$ $\mathbf{v}\bullet\mathbf{v} = 1$.

☐ Definition: If $\mathbf{w}$ is an arbitrary nonzero vector, then a **unit vector in the direction of** $\mathbf{w}$ is obtained by multiplying $\mathbf{w}$ by $||\mathbf{w}||^{-1}$: $\widehat{\mathbf{w}} = (1/||\mathbf{w}||)\mathbf{w}$ is a unit vector in the direction of $\mathbf{w}$. The caret mark over the vector will always be used to indicate a unit vector.

Examples: The standard basis vectors are all unit vectors. If

$$\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

then a unit vector in the direction of $\mathbf{w}$ is

$$\widehat{\mathbf{w}} = \frac{1}{||\mathbf{w}||}\mathbf{w} = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

☐ Definition: The process of replacing a vector $\mathbf{w}$ by a unit vector in its direction is called **normalizing** the vector.

For an arbitrary nonzero vector in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the corresponding unit vector is

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In physics and engineering courses, this particular vector is often denoted by $\widehat{\mathbf{r}}$. For instance, the gravitational force on a particle of mass m sitting at $(x, y, z)^t$ due to a particle of mass M sitting at the origin is

$$\mathbf{F} = \frac{-GMm}{r^2}\widehat{\mathbf{r}},$$

where $r^2 = x^2 + y^2 + z^2$.

18.2 Orthonormal bases

Although we know that any set of n linearly independent vectors in $\mathbb{R}^n$ can be used as a basis, there is a particularly nice collection of bases that we can use in Euclidean space.

□ Definition: A basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of $\mathbb{E}^n$ is said to be **orthonormal** if

1. $\mathbf{v}_i \bullet \mathbf{v}_j = 0,$ whenever $i \neq j$ — the vectors are mutually orthogonal, and
2. $\mathbf{v}_i \bullet \mathbf{v}_i = 1$ for all i — and they are all unit vectors.

Examples: The standard basis is orthonormal. The basis

$$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

is orthogonal, but not orthonormal. We can normalize these vectors to get the orthonormal basis

$$\left\{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}\right\}$$

You may recall that it can be tedious to compute the coordinates of a vector $\mathbf{w}$ in an arbitrary basis. One advantage of using an orthonormal basis is the following:

Theorem: Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis in $\mathbb{E}^n$. Let $\mathbf{w} \in \mathbb{E}^n$. Then

$$\mathbf{w} = (\mathbf{w} \bullet \mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w} \bullet \mathbf{v}_2)\mathbf{v}_2 + \cdots + (\mathbf{w} \bullet \mathbf{v}_n)\mathbf{v}_n.$$

That is, the i^{th} coordinate of $\mathbf{w}$ in this basis is given by $\mathbf{w}\bullet\mathbf{v}_i$, the dot product of $\mathbf{w}$ with the i^{th} basis vector. Alternatively, the coordinate vector of $\mathbf{w}$ in this orthonormal basis is

$$\mathbf{w}_v = \begin{pmatrix} \mathbf{w}\bullet\mathbf{v}_1 \\ \mathbf{w}\bullet\mathbf{v}_2 \\ \cdots \\ \mathbf{w}\bullet\mathbf{v}_n \end{pmatrix}.$$

Proof: Since we have a basis, we know there are unique numbers $c_1, \ldots, c_n$ (the coordinates of $\mathbf{w}$ in this basis) such that

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

Take the dot product of both sides of this equation with $\mathbf{v}_1$: using the linearity of the dot product, we get

$$\mathbf{v}_1\bullet\mathbf{w} = c_1(\mathbf{v}_1\bullet\mathbf{v}_1) + c_2(\mathbf{v}_1\bullet\mathbf{v}_2) + \cdots + c_n(\mathbf{v}_1\bullet\mathbf{v}_n).$$

Since the basis is orthonormal, all the dot products vanish except for the first, and we have $(\mathbf{v}_1\bullet\mathbf{w}) = c_1(\mathbf{v}_1\bullet\mathbf{v}_1) = c_1$. An identical argument holds for the general $\mathbf{v}_i$.

Example: Find the coordinates of the vector

$$\mathbf{w} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

in the basis

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \right\}.$$

Solution: $\mathbf{w}\bullet\mathbf{v}_1 = 2/\sqrt{2} - 3/\sqrt{2} = -1/\sqrt{2}$, and $\mathbf{w}\bullet\mathbf{v}_2 = 2/\sqrt{2} + 3/\sqrt{2} = 5/\sqrt{2}$. So the coordinates of $\mathbf{w}$ in this basis are

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

Exercises:

1. In $\mathbb{E}^2$, let
$$\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\} = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}.$$
Show that $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ is an orthonormal basis of $\mathbb{E}^2$ for any value of θ. What's the relation between $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$ and $\{\mathbf{i}, \mathbf{j}\} = \{\mathbf{e}_1(0), \mathbf{e}_2(0)\}$?

2. Let

$$\mathbf{v} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Find the coordinates of $\mathbf{v}$ in the basis $\{\mathbf{e}_1(\theta), \mathbf{e}_2(\theta)\}$

- By using the theorem above.
- By writing $\mathbf{v} = c_1\mathbf{e}_1(\theta) + c_2\mathbf{e}_2(\theta)$ and solving for $c_1,\ c_2$.
- By setting $E_\theta = (\mathbf{e}_1(\theta)|\mathbf{e}_2(\theta))$ and using the relation $\mathbf{v} = E_\theta \mathbf{v}_\theta$.