

54. Find the kernel of the linear transformation given in Exercise 50.

In Exercises 55 and 56, find the image of  $\mathbf{v} = (i, i)$  for the indicated composition, where  $T_1$  and  $T_2$  are given by the following matrices.

$$A_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}$$

55.  $T_2 \circ T_1$

56.  $T_1 \circ T_2$

57. Determine which of the following sets are subspaces of the vector space of  $2 \times 2$  complex matrices.

- The set of  $2 \times 2$  symmetric matrices.
- The set of  $2 \times 2$  matrices  $A$  satisfying  $(\bar{A})^T = A$ .
- The set of  $2 \times 2$  matrices in which all entries are real.
- The set of  $2 \times 2$  diagonal matrices.

58. Determine which of the following sets are subspaces of the vector space of complex-valued functions (see Example 4).

- The set of all functions  $f$  satisfying  $f(i) = 0$ .
- The set of all functions  $f$  satisfying  $f(0) = 1$ .
- The set of all functions  $f$  satisfying  $f(i) = f(-i)$ .

## 8.5 UNITARY AND HERMITIAN MATRICES

Problems involving diagonalization of complex matrices, and the associated eigenvalue problems, require the concept of **unitary** and **Hermitian** matrices. These matrices roughly correspond to orthogonal and symmetric real matrices. In order to define unitary and Hermitian matrices, we first introduce the concept of the **conjugate transpose** of a complex matrix.

### Definition of the Conjugate Transpose of a Complex Matrix

The **conjugate transpose** of a complex matrix  $A$ , denoted by  $A^*$ , is given by

$$A^* = \bar{A}^T$$

where the entries of  $\bar{A}$  are the complex conjugates of the corresponding entries of  $A$ .

Note that if  $A$  is a matrix with real entries, then  $A^* = A^T$ . To find the conjugate transpose of a matrix, we first calculate the complex conjugate of each entry and then take the transpose of the matrix, as shown in the following example.

#### EXAMPLE 1 Finding the Conjugate Transpose of a Complex Matrix

Determine  $A^*$  for the matrix

$$A = \begin{bmatrix} 3 + 7i & 0 \\ 2i & 4 - i \end{bmatrix}.$$

Solution

$$\bar{A} = \begin{bmatrix} \overline{3+7i} & \bar{0} \\ \overline{2i} & \overline{4-i} \end{bmatrix} = \begin{bmatrix} 3-7i & 0 \\ -2i & 4+i \end{bmatrix}$$

$$A^* = \bar{A}^T = \begin{bmatrix} 3-7i & -2i \\ 0 & 4+i \end{bmatrix}$$

We list several properties of the conjugate transpose of a matrix in the following theorem. The proofs of these properties are straightforward and are left for you to supply in Exercises 49–52.

**Theorem 8.8**

Properties of  
Conjugate Transpose

If  $A$  and  $B$  are complex matrices and  $k$  is a complex number, then the following properties are true.

1.  $(A^*)^* = A$
2.  $(A + B)^* = A^* + B^*$
3.  $(kA)^* = \bar{k}A^*$
4.  $(AB)^* = B^*A^*$

## Unitary Matrices

Recall that a real matrix  $A$  is *orthogonal* if and only if  $A^{-1} = A^T$ . In the complex system, matrices having the property that  $A^{-1} = A^*$  are more useful and we call such matrices **unitary**.

### Definition of a Unitary Matrix

A complex matrix  $A$  is called **unitary** if

$$A^{-1} = A^*.$$

#### EXAMPLE 2 *A Unitary Matrix*

Show that the following matrix is unitary.

$$A = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$

**Solution** Since

$$AA^* = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I_2$$

we conclude that  $A^* = A^{-1}$ . Therefore,  $A$  is a unitary matrix.

In Section 7.3, we showed that a real matrix is orthogonal if and only if its row (or column) vectors form an orthonormal set. For complex matrices, this property characterizes matrices that are unitary. Note that we call a set of vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$$

in  $C^n$  (complex Euclidean space) orthonormal if the following are true.

1.  $\|\mathbf{v}_i\| = 1, i = 1, 2, \dots, m$
2.  $\mathbf{v}_i \cdot \mathbf{v}_j = 0, i \neq j$

The proof of the following theorem is similar to the proof of Theorem 7.8 given in Section 7.3.

### Theorem 8.9

#### Unitary Matrices

An  $n \times n$  complex matrix  $A$  is unitary if and only if its row (or column) vectors form an orthonormal set in  $C^n$ .

#### EXAMPLE 3 The Row Vectors of a Unitary Matrix

Show that the following complex matrix is unitary by showing that its set of row vectors form an orthonormal set in  $C^3$ .

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1+i}{2} & -\frac{1}{2} \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{5i}{2\sqrt{15}} & \frac{3+i}{2\sqrt{15}} & \frac{4+3i}{2\sqrt{15}} \end{bmatrix}$$

**Solution** We let  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $\mathbf{r}_3$  be defined as follows.

$$\mathbf{r}_1 = \left( \frac{1}{2}, \frac{1+i}{2}, -\frac{1}{2} \right)$$

$$\mathbf{r}_2 = \left( -\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\mathbf{r}_3 = \left( \frac{5i}{2\sqrt{15}}, \frac{3+i}{2\sqrt{15}}, \frac{4+3i}{2\sqrt{15}} \right)$$

The length of  $\mathbf{r}_1$  is

$$\begin{aligned} \|\mathbf{r}_1\| &= (\mathbf{r}_1 \cdot \mathbf{r}_1)^{1/2} \\ &= \left[ \left( \frac{1}{2} \right) \overline{\left( \frac{1}{2} \right)} + \left( \frac{1+i}{2} \right) \overline{\left( \frac{1+i}{2} \right)} + \left( -\frac{1}{2} \right) \overline{\left( -\frac{1}{2} \right)} \right]^{1/2} \\ &= \left[ \frac{1}{4} + \frac{2}{4} + \frac{1}{4} \right]^{1/2} = 1. \end{aligned}$$

The vectors  $\mathbf{r}_2$  and  $\mathbf{r}_3$  can also be shown to be unit vectors. The inner product of  $\mathbf{r}_1$  and  $\mathbf{r}_2$  is given by

$$\begin{aligned}\mathbf{r}_1 \cdot \mathbf{r}_2 &= \left(\frac{1}{2}\right)\overline{\left(\frac{-i}{\sqrt{3}}\right)} + \left(\frac{1+i}{2}\right)\overline{\left(\frac{i}{\sqrt{3}}\right)} + \left(\frac{-1}{2}\right)\overline{\left(\frac{1}{\sqrt{3}}\right)} \\ &= \left(\frac{1}{2}\right)\left(\frac{i}{\sqrt{3}}\right) + \left(\frac{1+i}{2}\right)\left(\frac{-i}{\sqrt{3}}\right) + \left(\frac{-1}{2}\right)\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{i}{2\sqrt{3}} - \frac{i}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} - \frac{1}{2\sqrt{3}} \\ &= 0.\end{aligned}$$

Similarly,  $\mathbf{r}_1 \cdot \mathbf{r}_3 = 0$  and  $\mathbf{r}_2 \cdot \mathbf{r}_3 = 0$  and we can conclude that  $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$  is an orthonormal set. (Try showing that the column vectors of  $A$  also form an orthonormal set in  $C^3$ .)

## Hermitian Matrices

A real matrix is called symmetric if it is equal to its own transpose. In the complex system, the more useful type of matrix is one that is equal to its own *conjugate* transpose. We call such a matrix **Hermitian** after the French mathematician Charles Hermite (1822–1901).

### Definition of a Hermitian Matrix

A square matrix  $A$  is **Hermitian** if

$$A = A^*.$$

As with symmetric matrices, we can easily recognize Hermitian matrices by inspection. To see this, consider the  $2 \times 2$  matrix.

$$A = \begin{bmatrix} a_1 + a_2i & b_1 + b_2i \\ c_1 + c_2i & d_1 + d_2i \end{bmatrix}.$$

The conjugate transpose of  $A$  has the form

$$\begin{aligned}A^* &= \overline{A}^T \\ &= \begin{bmatrix} \overline{a_1 + a_2i} & \overline{c_1 + c_2i} \\ \overline{b_1 + b_2i} & \overline{d_1 + d_2i} \end{bmatrix} \\ &= \begin{bmatrix} a_1 - a_2i & c_1 - c_2i \\ b_1 - b_2i & d_1 - d_2i \end{bmatrix}.\end{aligned}$$

If  $A$  is Hermitian, then  $A = A^*$  and we can conclude that  $A$  must be of the form

$$A = \begin{bmatrix} a_1 & b_1 + b_2i \\ b_1 - b_2i & d_1 \end{bmatrix}.$$

Similar results can be obtained for Hermitian matrices of order  $n \times n$ . In other words, a square matrix  $A$  is Hermitian if and only if the following two conditions are met.

1. The entries on the main diagonal of  $A$  are real.
2. The entry  $a_{ij}$  in the  $i$ th row and the  $j$ th column is the complex conjugate of the entry  $a_{ji}$  in the  $j$ th row and  $i$ th column.

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**EXAMPLE 4** *Hermitian Matrices*

Which of the following matrices are Hermitian?

(a)  $\begin{bmatrix} 1 & 3 - i \\ 3 + i & i \end{bmatrix}$

(b)  $\begin{bmatrix} 0 & 3 - 2i \\ 3 - 2i & 4 \end{bmatrix}$

(c)  $\begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$

(d)  $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix}$

- Solution**
- (a) This matrix is not Hermitian because it has an imaginary entry on its main diagonal.  
 (b) This matrix is symmetric but not Hermitian because the entry in the first row and second column is not the complex conjugate of the entry in the second row and first column.  
 (c) This matrix is Hermitian.  
 (d) This matrix is Hermitian, because all real symmetric matrices are Hermitian.

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One of the most important characteristics of Hermitian matrices is that their eigenvalues are real. This is formally stated in the next theorem.

**Theorem 8.10**

The Eigenvalues of a Hermitian Matrix

If  $A$  is a Hermitian matrix, then its eigenvalues are real numbers.

**Proof** Let  $\lambda$  be an eigenvalue of  $A$  and

$$\mathbf{v} = \begin{bmatrix} a_1 + b_1i \\ a_2 + b_2i \\ \vdots \\ a_n + b_ni \end{bmatrix}$$

be its corresponding eigenvector. If we multiply both sides of the equation  $A\mathbf{v} = \lambda\mathbf{v}$  by the row vector  $\mathbf{v}^*$ , we obtain

$$\mathbf{v}^*A\mathbf{v} = \mathbf{v}^*(\lambda\mathbf{v}) = \lambda(\mathbf{v}^*\mathbf{v}) = \lambda(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots + a_n^2 + b_n^2).$$

Furthermore, since

$$(\mathbf{v}^*A\mathbf{v})^* = \mathbf{v}^*A^*(\mathbf{v}^*)^* = \mathbf{v}^*A\mathbf{v}$$

it follows that  $\mathbf{v}^*A\mathbf{v}$  is a Hermitian  $1 \times 1$  matrix. This implies that  $\mathbf{v}^*A\mathbf{v}$  is a real number, and we may conclude that  $\lambda$  is real.

REMARK: Note that this theorem implies that the eigenvalues of a *real symmetric matrix* are real, as stated in Theorem 7.7.

To find the eigenvalues of complex matrices, we follow the same procedure as for real matrices.

**EXAMPLE 5** *Finding the Eigenvalues of a Hermitian Matrix*

Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$$

**Solution** The characteristic polynomial of  $A$  is

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - 3 & -2 + i & 3i \\ -2 - i & \lambda & -1 + i \\ -3i & -1 - i & \lambda \end{vmatrix} \\ &= (\lambda - 3)(\lambda^2 - 2) - (-2 + i)[(-2 - i)\lambda - (3i + 3)] \\ &\quad + 3i[(1 + 3i) + 3\lambda i] \\ &= (\lambda^3 - 3\lambda^2 - 2\lambda + 6) - (5\lambda + 9 + 3i) + (3i - 9 - 9\lambda) \\ &= \lambda^3 - 3\lambda^2 - 16\lambda - 12 \\ &= (\lambda + 1)(\lambda - 6)(\lambda + 2) \end{aligned}$$

which implies that the eigenvalues of  $A$  are  $-1$ ,  $6$ , and  $-2$ .

To find the eigenvectors of a complex matrix, we use a similar procedure to that used for a real matrix. For instance, in Example 5, the eigenvector corresponding to the eigenvalue  $\lambda = -1$  is obtained by solving the following equation.

$$\begin{bmatrix} \lambda - 3 & -2 + i & 3i \\ -2 - i & \lambda & -1 + i \\ -3i & -1 - i & \lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 + i & 3i \\ -2 - i & -1 & -1 + i \\ -3i & -1 - i & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Gauss-Jordan elimination, or a computer or calculator, we obtain the following eigenvector corresponding to  $\lambda_1 = -1$ .

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 + 2i \\ 1 \end{bmatrix} \quad (\lambda_1 = -1)$$

Eigenvectors for  $\lambda_2 = 6$  and  $\lambda_3 = -2$  can be found in a similar manner. They are

$$\begin{bmatrix} 1 - 21i \\ 6 - 9i \\ 13 \end{bmatrix} \quad (\lambda_2 = 6); \quad \begin{bmatrix} 1 + 3i \\ -2 - i \\ 5 \end{bmatrix} \quad (\lambda_3 = -2)$$

### TECHNOLOGY NOTE

Some computers and calculators have built-in programs for finding the eigenvalues and corresponding eigenvectors of complex matrices. For example, on the TI-85, the eigVI key on the MATRX MATH menu calculates the eigenvalues of the matrix  $A$ , and the eigVc key gives the corresponding eigenvectors.

Just as we saw in Section 7.3 that real symmetric matrices were orthogonally diagonalizable, we will show now that Hermitian matrices are **unitarily diagonalizable**. A square matrix  $A$  is unitarily diagonalizable if there exists a unitary matrix  $P$  such that

$$P^{-1}AP$$

is a diagonal matrix. Since  $P$  is unitary,  $P^{-1} = P^*$ , so an equivalent statement is that  $A$  is unitarily diagonalizable if there exists a unitary matrix  $P$  such that  $P^*AP$  is a diagonal matrix. The next theorem tells us that Hermitian matrices are unitarily diagonalizable.

#### Theorem 8.11

#### Hermitian Matrices and Diagonalization

If  $A$  is an  $n \times n$  Hermitian matrix, then

1. eigenvectors corresponding to distinct eigenvalues are orthogonal.
2.  $A$  is unitarily diagonalizable.

**Proof** To prove part 1, let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be two eigenvectors corresponding to the distinct (and real) eigenvalues  $\lambda_1$  and  $\lambda_2$ . Because  $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$  and  $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ , we have the following equations for the matrix product  $(A\mathbf{v}_1)^*\mathbf{v}_2$ .

$$(A\mathbf{v}_1)^*\mathbf{v}_2 = \mathbf{v}_1^*A^*\mathbf{v}_2 = \mathbf{v}_1^*A\mathbf{v}_2 = \mathbf{v}_1^*\lambda_2\mathbf{v}_2 = \lambda_2\mathbf{v}_1^*\mathbf{v}_2$$

$$(A\mathbf{v}_1)^*\mathbf{v}_2 = (\lambda_1\mathbf{v}_1)^*\mathbf{v}_2 = \mathbf{v}_1^*\lambda_1\mathbf{v}_2 = \lambda_1\mathbf{v}_1^*\mathbf{v}_2$$

Therefore,

$$\lambda_2\mathbf{v}_1^*\mathbf{v}_2 - \lambda_1\mathbf{v}_1^*\mathbf{v}_2 = 0$$

$$(\lambda_2 - \lambda_1)\mathbf{v}_1^*\mathbf{v}_2 = 0$$

$$\mathbf{v}_1^*\mathbf{v}_2 = 0 \quad \text{since } \lambda_1 \neq \lambda_2,$$

and we have shown that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are orthogonal. Part 2 of Theorem 8.11 is often called the **Spectral Theorem**, and its proof is omitted.

**EXAMPLE 6** *The Eigenvectors of a Hermitian Matrix*

The eigenvectors of the Hermitian matrix given in Example 5 are mutually orthogonal because the eigenvalues are distinct. We can verify this by calculating the Euclidean inner products  $\mathbf{v}_1 \cdot \mathbf{v}_2$ ,  $\mathbf{v}_1 \cdot \mathbf{v}_3$  and  $\mathbf{v}_2 \cdot \mathbf{v}_3$ . For example,

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= (-1)\overline{(1 - 21i)} + (1 + 2i)\overline{(6 - 9i)} + (1)\overline{(13)} \\ &= (-1)(1 + 21i) + (1 + 2i)(6 + 9i) + 13 \\ &= -1 - 21i + 6 + 12i + 9i - 18 + 13 \\ &= 0.\end{aligned}$$

The other two inner products  $\mathbf{v}_1 \cdot \mathbf{v}_3$  and  $\mathbf{v}_2 \cdot \mathbf{v}_3$  can be shown to equal zero in a similar manner.

The three eigenvectors in Example 6 are mutually orthogonal because they correspond to distinct eigenvalues of the Hermitian matrix  $A$ . Two or more eigenvectors corresponding to the same eigenvalue may not be orthogonal. However, once we obtain any set of linearly independent eigenvectors for a given eigenvalue, we can use the Gram-Schmidt orthonormalization process to obtain an orthogonal set.

**EXAMPLE 7** *Diagonalization of a Hermitian Matrix*

Find a unitary matrix  $P$  such that  $P^*AP$  is a diagonal matrix where

$$A = \begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}.$$

**Solution** The eigenvectors of  $A$  are given after Example 5. We form the matrix  $P$  by normalizing these three eigenvectors and using the results to create the columns of  $P$ . Thus, since

$$\begin{aligned}\|\mathbf{v}_1\| &= \|(-1, 1 + 2i, 1)\| = \sqrt{1 + 5 + 1} = \sqrt{7} \\ \|\mathbf{v}_2\| &= \|(1 - 21i, 6 - 9i, 13)\| = \sqrt{442 + 117 + 169} = \sqrt{728} \\ \|\mathbf{v}_3\| &= \|(1 + 3i, -2 - i, 5)\| = \sqrt{10 + 5 + 25} = \sqrt{40}\end{aligned}$$

we obtain the unitary matrix  $P$ ,

$$P = \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{1 - 21i}{\sqrt{728}} & \frac{1 + 3i}{\sqrt{40}} \\ \frac{1 + 2i}{\sqrt{7}} & \frac{6 - 9i}{\sqrt{728}} & \frac{-2 - i}{\sqrt{40}} \\ \frac{1}{\sqrt{7}} & \frac{13}{\sqrt{728}} & \frac{5}{\sqrt{40}} \end{bmatrix}.$$



Try computing the product  $P^*AP$  for the matrices  $A$  and  $P$  in Example 7 to see that you obtain

$$P^*AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

where  $-1$ ,  $6$ , and  $-2$  are the eigenvalues of  $A$ .

We have seen that Hermitian matrices are unitarily diagonalizable. However, it turns out that there is a larger class of matrices, called **normal** matrices, which are also unitarily diagonalizable. A square complex matrix  $A$  is **normal** if it commutes with its conjugate transpose:  $AA^* = A^*A$ . The main theorem of normal matrices says that a complex matrix  $A$  is normal if and only if it is unitarily diagonalizable. You are asked to explore normal matrices further in Exercise 59.

The properties of complex matrices described in this section are comparable to the properties of real matrices discussed in Chapter 7. The following summary indicates the correspondence between unitary and Hermitian complex matrices when compared with orthogonal and symmetric real matrices.

### Comparison of Hermitian and Symmetric Matrices

*A is a symmetric matrix  
(Real)*

1. Eigenvalues of  $A$  are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There exists an orthogonal matrix  $P$  such that

$$P^TAP$$

is diagonal.

*A is a Hermitian matrix  
(Complex)*

1. Eigenvalues of  $A$  are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. There exists a unitary matrix  $P$  such that

$$P^*AP$$

is diagonal.

## SECTION 8.5 EXERCISES

In Exercises 1–8, determine the conjugate transpose of the given matrix.

1.  $A = \begin{bmatrix} i & -i \\ 2 & 3i \end{bmatrix}$

2.  $A = \begin{bmatrix} 1 + 2i & 2 - i \\ 1 & 1 \end{bmatrix}$

3.  $A = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

4.  $A = \begin{bmatrix} 4 + 3i & 2 + i \\ 2 - i & 6i \end{bmatrix}$

5.  $A = \begin{bmatrix} 0 & 5 + i & \sqrt{2}i \\ 5 - i & 6 & 4 \\ -\sqrt{2}i & 4 & 3 \end{bmatrix}$

6.  $A = \begin{bmatrix} 2 + i & 3 - i & 4 + 5i \\ 3 - i & 2 & 6 - 2i \end{bmatrix}$

7.  $A = \begin{bmatrix} 7 + 5i \\ 2i \\ 4 \end{bmatrix}$

8.  $A = \begin{bmatrix} 2 & i \\ 5 & 3i \\ 0 & 6 - i \end{bmatrix}$

In Exercises 9–12, explain why the given matrix is *not* unitary.

9.  $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$

10.  $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

11.  $A = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$

12.  $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1+i}{2} \\ -\frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1+i}{2} \end{bmatrix}$

In Exercises 13–18, determine whether  $A$  is unitary by calculating  $AA^*$ .

13.  $A = \begin{bmatrix} 1 + i & 1 + i \\ 1 - i & 1 - i \end{bmatrix}$

14.  $A = \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix}$

15.  $A = I_n$

16.  $A = \begin{bmatrix} i & i \\ \sqrt{2} & \sqrt{2} \\ i & -i \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}$

17.  $A = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{6}} \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{6}} \end{bmatrix}$

18.  $A = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{5}{5} & \frac{5}{5} \\ \frac{3}{5}i & \frac{4}{5}i \\ \frac{5}{5} & \frac{5}{5} \end{bmatrix}$

In Exercises 19–22, (a) verify that  $A$  is unitary by showing that its rows are orthonormal, and (b) determine the inverse of  $A$ .

19.  $A = \begin{bmatrix} \frac{4}{5} & \frac{3}{5}i \\ \frac{3}{5} & \frac{4}{5}i \\ \frac{5}{5} & \frac{5}{5}i \end{bmatrix}$

20.  $A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

21.  $A = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{3} - i & 1 + \sqrt{3}i \\ \sqrt{3} + i & 1 - \sqrt{3}i \end{bmatrix}$

22.  $A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1+i}{\sqrt{6}} & 0 & \frac{1-i}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$

In Exercises 23–28, determine whether the matrix  $A$  is Hermitian.

23.  $A = \begin{bmatrix} 0 & 2 + i & 1 \\ 2 - i & i & 0 \\ 1 & 0 & 1 \end{bmatrix}$

24.  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

25.  $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

26.  $A = \begin{bmatrix} 1 & 2 + i & 3 - i \\ 2 - i & 2 & 3 + i \end{bmatrix}$

27.  $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

28.  $A = \begin{bmatrix} 1 & \sqrt{2} + i & 5 \\ \sqrt{2} - i & 2 & 3 + i \\ 5 & 3 - i & 6 \end{bmatrix}$

In Exercises 29–34, determine the eigenvalues of the matrix  $A$ .

$$29. A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad 30. A = \begin{bmatrix} 0 & 2+i \\ 2-i & 4 \end{bmatrix}$$

$$31. A = \begin{bmatrix} 3 & 1-i \\ 1+i & 2 \end{bmatrix} \quad 32. A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}$$

$$33. A = \begin{bmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$$

$$34. A = \begin{bmatrix} 1 & 4 & 1-i \\ 0 & i & 3i \\ 0 & 0 & 2+i \end{bmatrix}$$

In Exercises 35–38, determine the eigenvectors of the given matrix.

35. The matrix in Exercise 29.

36. The matrix in Exercise 30.

37. The matrix in Exercise 33.

38. The matrix in Exercise 32.

In Exercises 39–43, find a unitary matrix  $P$  that diagonalizes the given matrix  $A$ .

$$39. A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad 40. A = \begin{bmatrix} 0 & 2+i \\ 2-i & 4 \end{bmatrix}$$

$$41. A = \begin{bmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$$

$$42. A = \begin{bmatrix} 4 & 2+2i \\ 2-2i & 6 \end{bmatrix}$$

$$43. A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1+i \\ 0 & -1-i & 0 \end{bmatrix}$$

44. Let  $z$  be a complex number with modulus 1. Show that the following matrix is unitary.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} z & \bar{z} \\ iz & -i\bar{z} \end{bmatrix}$$

In Exercises 45–48, use the result of Exercise 44 to determine  $a$ ,  $b$ , and  $c$  so that  $A$  is unitary.

$$45. A = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & a \\ b & c \end{bmatrix} \quad 46. A = \frac{1}{\sqrt{2}} \begin{bmatrix} 3-4i & a \\ 5 & c \end{bmatrix}$$

$$47. A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & a \\ b & c \end{bmatrix} \quad 48. A = \frac{1}{\sqrt{2}} \begin{bmatrix} a & \frac{6+3i}{\sqrt{45}} \\ b & c \end{bmatrix}$$

In Exercises 49–52, prove the given formula, where  $A$  and  $B$  are  $n \times n$  complex matrices.

$$49. (A^*)^* = A \quad 50. (A+B)^* = A^* + B^*$$

$$51. (kA)^* = \bar{k}A^* \quad 52. (AB)^* = B^*A^*$$

53. Let  $A$  be a matrix such that  $A^* + A = O$ . Prove that  $iA$  is Hermitian.

54. Show that  $\det(\bar{A}) = \overline{\det(A)}$ , where  $A$  is a  $2 \times 2$  matrix.

In Exercises 55–56, assume that the result of Exercise 54 is true for matrices of any size.

55. Show that  $\det(A^*) = \overline{\det(A)}$ .

56. Prove that if  $A$  is unitary, then  $|\det(A)| = 1$ .

57. (a) Prove that every Hermitian matrix  $A$  can be written as the sum  $A = B + iC$ , where  $B$  is a real symmetric matrix and  $C$  is real and skew-symmetric.

(b) Use part (a) to write the matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

as a sum  $A = B + iC$ , where  $B$  is a real symmetric matrix and  $C$  is real and skew-symmetric.

(c) Prove that every  $n \times n$  complex matrix  $A$  can be written as  $A = B + iC$ , where  $B$  and  $C$  are Hermitian.

(d) Use part (c) to write the complex matrix

$$A = \begin{bmatrix} i & 2 \\ 2+i & 1-2i \end{bmatrix}$$

as a sum  $A = B + iC$ , where  $B$  and  $C$  are Hermitian.

58. Determine which of the following sets are subspaces of the vector space of  $n \times n$  complex matrices.
- The set of  $n \times n$  Hermitian matrices.
  - The set of  $n \times n$  unitary matrices.
  - The set of  $n \times n$  normal matrices.
59. (a) Prove that every Hermitian matrix is normal.  
 (b) Prove that every unitary matrix is normal.  
 (c) Find a  $2 \times 2$  matrix that is Hermitian, but not unitary.
- (d) Find a  $2 \times 2$  matrix that is unitary, but not Hermitian.  
 (e) Find a  $2 \times 2$  matrix that is normal, but neither Hermitian nor unitary.  
 (f) Find the eigenvalues and corresponding eigenvectors of your matrix from part (e).  
 (g) Show that the complex matrix

$$\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$$

is not diagonalizable. Is this matrix normal?

## CHAPTER 8 REVIEW EXERCISES

In Exercises 1–6, perform the given operation.

- Find  $u + z$ :  $u = 2 - 4i$ ,  $z = 4i$
- Find  $u - z$ :  $u = 4$ ,  $z = 8i$
- Find  $uz$ :  $u = 4 - 2i$ ,  $z = 4 + 2i$
- Find  $uz$ :  $u = 2i$ ,  $z = 1 - 2i$
- Find  $\frac{u}{z}$ :  $u = 6 - 2i$ ,  $z = 3 - 3i$
- Find  $\frac{u}{z}$ :  $u = 7 + i$ ,  $z = i$

In Exercises 7–10, find all zeros of the given polynomial.

- $x^2 - 4x + 8$
- $x^2 - 4x + 7$
- $3x^2 + 3x + 3$
- $x^3 + 2x^2 + 2x + 1$

In Exercises 11–14, perform the given operation using

$$A = \begin{bmatrix} 4 - i & 2 \\ 3 & 3 + i \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 + i & i \\ 2i & 2 + i \end{bmatrix}.$$

- $A + B$
- $2iB$
- $\det(A - B)$
- $3BA$

In Exercises 15–20, perform the given operation using  $w = 2 - 2i$ ,  $v = 3 + i$ , and  $z = -1 + 2i$ .

- $\bar{z}$
- $\bar{v}$
- $|w|$
- $|v\bar{z}|$
- $\frac{w}{v}$
- $\frac{z}{w}$

In Exercises 21–24, perform the indicated operation.

- $\frac{2 + i}{2 - i}$
- $\frac{1 + i}{-1 + 2i}$

$$23. \frac{(1 - 2i)(1 + 2i)}{3 - 3i} \quad 24. \frac{5 + 2i}{(-2 + 2i)(2 - 3i)}$$

In Exercises 25 and 26, find  $A^{-1}$  (if it exists).

$$25. A = \begin{bmatrix} 3 - i & -1 - 2i \\ -\frac{23}{5} + \frac{11}{5}i & 2 + 3i \end{bmatrix}$$

$$26. A = \begin{bmatrix} 5 & 1 - i \\ 0 & i \end{bmatrix}$$

In Exercises 27–30, determine the polar form of the complex number.

- $4 + 4i$
- $3 + 2i$
- $7 - 4i$
- $\sqrt{3} + i$

In Exercises 31–34, find the standard form of the given complex number.

$$31. 5 \left[ \cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right]$$

$$32. 4 \left( \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right)$$

$$33. 6 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \quad 34. 7 \left( \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$$

In Exercises 35–38, perform the indicated operation. Leave the result in polar form.

$$35. \left[ 4 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right] \left[ 3 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]$$

$$36. \left[ \frac{1}{2} \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \right] \left[ 2 \left( \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right) \right]$$

$$37. \frac{9[\cos(\pi/2) + i \sin(\pi/2)]}{6[\cos(2\pi/3) + i \sin(2\pi/3)]}$$

$$38. \frac{4[\cos(\pi/4) + i \sin(\pi/4)]}{7[\cos(\pi/3) + i \sin(\pi/3)]}$$

In Exercises 39–42, find the indicated power of the given number and express the result in polar form.

$$39. (-1 - i)^4$$

$$40. (2i)^3$$

$$41. \left[ \sqrt{2} \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right]^7$$

$$42. \left[ 5 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \right]^4$$

In Exercises 43–46, express the given roots in standard form.

$$43. \text{Square roots: } 25 \left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$44. \text{Cube roots: } 27 \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$45. \text{Cube roots: } i$$

$$46. \text{Fourth roots: } 16 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

In Exercises 47 and 48, determine the conjugate transpose of the given matrix.

$$47. A = \begin{bmatrix} -1 + 4i & 3 + i \\ 3 - i & 2 + i \end{bmatrix}$$

$$48. A = \begin{bmatrix} 5 & 2 - i & 3 + 2i \\ 2 + 2i & 3 - 2i & i \\ 3i & 2 + i & -1 - 2i \end{bmatrix}$$

In Exercises 49–52, find the indicated vector using  $\mathbf{u} = (4i, 2 + i)$ ,  $\mathbf{v} = (3, -i)$ , and  $\mathbf{w} = (3 - i, 4 + i)$ .

$$49. 7\mathbf{u} - \mathbf{v}$$

$$50. 3i\mathbf{w} + (4 - i)\mathbf{v}$$

$$51. i\mathbf{u} + i\mathbf{v} - i\mathbf{w}$$

$$52. (3 + 2i)\mathbf{u} - (-2i)\mathbf{w}$$

In Exercises 53 and 54, determine the Euclidean norm of the given vector.

$$53. \mathbf{v} = (3 - 5i, 2i)$$

$$54. \mathbf{v} = (3i, -1 - 5i, 3 + 2i)$$

In Exercises 55 and 56, find the Euclidean distance between the given vectors.

$$55. \mathbf{v} = (2 - i, i), \quad \mathbf{u} = (i, 2 - i)$$

$$56. \mathbf{v} = (2 + i, -1 + 2i, 3i), \quad \mathbf{u} = (4 - 2i, 3 + 2i, 4)$$

In Exercises 57–60, determine whether the given matrix is unitary.

$$57. \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$58. \begin{bmatrix} \frac{2+i}{4} & \frac{1+i}{4} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$$

$$59. \begin{bmatrix} 1 & 0 \\ i & -i \end{bmatrix}$$

$$60. \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & i & 0 \\ \frac{1+i}{2} & 0 & \frac{-1-i}{2} \end{bmatrix}$$

In Exercises 61 and 62, determine whether the given matrix is Hermitian.

$$61. \begin{bmatrix} 1 & -1 + i & 2 - i \\ 1 - i & 3 & i \\ 2 + i & -i & 4 \end{bmatrix}$$

$$62. \begin{bmatrix} 9 & 2 - i & 2 \\ 2 + i & 0 & -1 - i \\ 2 & -1 + i & 3 \end{bmatrix}$$

In Exercises 63 and 64, find the eigenvalues and corresponding eigenvectors of the given matrix.

$$63. \begin{bmatrix} 4 & 2 - i \\ 2 + i & 0 \end{bmatrix}$$

$$64. \begin{bmatrix} 2 & 0 & -i \\ 0 & 3 & 0 \\ i & 0 & 2 \end{bmatrix}$$

65. Prove that if  $A$  is an invertible matrix, then  $A^*$  is also invertible.

66. Determine all complex numbers  $z$  such that  $z = -\bar{z}$ .

67. Prove that if the product of two complex numbers is zero, then one of the numbers must be zero.

68. (a) Find the determinant of the following Hermitian matrix.

$$\begin{bmatrix} 3 & 2 - i & -3i \\ 2 + i & 0 & 1 - i \\ 3i & 1 + i & 0 \end{bmatrix}$$

(b) Prove that the determinant of any Hermitian matrix is real.

69. Let  $A$  and  $B$  be Hermitian matrices. Prove that  $AB = BA$  if and only if  $AB$  is Hermitian.

70. Let  $\mathbf{u}$  be a unit vector in  $C^n$ . Define  $H = I - 2\mathbf{u}\mathbf{u}^*$ . Prove that  $H$  is an  $n \times n$  Hermitian and unitary matrix.

71. Use mathematical induction to prove DeMoivre's Theorem.
72. Prove that if  $z$  is a zero of a polynomial equation with real coefficients, then the conjugate of  $z$  is also a zero.
73. Show that if  $z_1 + z_2$  and  $z_1\overline{z_2}$  are both nonzero real numbers, then  $z_1$  and  $z_2$  are both real numbers.
74. Prove that if  $z$  and  $w$  are complex numbers, then  $|z + w| \leq |z| + |w|$ .
75. Prove that for all vectors  $\mathbf{u}$  and  $\mathbf{v}$  in a complex inner product space,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} [\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 + i\|\mathbf{u} + i\mathbf{v}\|^2 - i\|\mathbf{u} - i\mathbf{v}\|^2].$$

## CHAPTER 8 PROJECTS

### 1 Population Growth and Dynamical Systems - II

In the projects for Chapter 7, you were asked to model the population of two species using a system of differential equations of the form

$$\begin{aligned} y_1'(t) &= ay_1(t) + by_2(t) \\ y_2'(t) &= cy_1(t) + dy_2(t). \end{aligned}$$

The constants  $a$ ,  $b$ ,  $c$ , and  $d$  depend on the particular species being studied. In Chapter 7, we looked at an example of a predator–prey relationship, in which  $a = 0.5$ ,  $b = 0.6$ ,  $c = -0.4$ , and  $d = 3.0$ . Suppose we now consider a slightly different model.

$$\begin{aligned} y_1'(t) &= 0.6y_1(t) + 0.8y_2(t), & y_1(0) &= 36 \\ y_2'(t) &= -0.8y_1(t) + 0.6y_2(t), & y_2(0) &= 121 \end{aligned}$$

1. Use the diagonalization technique to find the general solutions  $y_1(t)$  and  $y_2(t)$  at any time  $t > 0$ . Although the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

are complex, the same principles apply, and you can obtain complex exponential solutions.

2. Convert your complex solutions to real solutions by observing that if  $\lambda = a + bi$  is a (complex) eigenvalue of  $A$  with (complex) eigenvector  $\mathbf{v}$ , then the real and imaginary parts of  $e^{\lambda t}\mathbf{v}$  form a linearly independent pair of (real) solutions. You will need to use the formula  $e^{i\theta} = \cos \theta + i \sin \theta$ .
3. Use the initial conditions to find the explicit form of the (real) solutions to the original equations.
4. If you have access to a computer or graphing calculator, graph the solutions obtained in part (3) over the domain  $0 \leq t \leq 3$ . At what moment are the two populations equal?
5. Interpret the solution in terms of the long-term population trend for the two species. Does one species ultimately disappear? Why or why not? Contrast this solution to that obtained for the model in Chapter 7.
6. If you have access to a computer or graphing calculator that can numerically solve differential equations, use it to graph the solutions to the original system of equations. Does this numerical approximation appear to be accurate?