Midterm Exam I- Solutions

Problem 1. *(20 pts) We know the following facts about events A and B:*

- *The probability of event A is 0.25.*
- *Event A and event B are independent.*
- *The probability that neither A nor B occurs is twelve times the probability that both A and B occur.*

Find the probabilities $\mathbb{P}(B)$, $\mathbb{P}(A \cap B)$, $\mathbb{P}(A' \cup B)$, and the conditional probabilities $\mathbb{P}(B|A')$, $\mathbb{P}(A \cap B|A \cup B)$. **(Hint: You can use a Venn diagram.)**

Solution. If we define

$$a = \mathbb{P}(A \cap B'); \quad b = \mathbb{P}(B \cap A'); \quad c = \mathbb{P}(A \cap B); \quad d = \mathbb{P}(A' \cap B'),$$

then $a + b + c + d = 1$ and we are given that

- $\mathbb{P}(A) = a + c = 0.25.$
- $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, that is $c = (a + c)(b + c)$.
- $\mathbb{P}(\mathbb{A}' \cap \mathbb{B}') = 12\mathbb{P}(A \cap B)$, that is $d = 12c$.

Solving for a, b, c, d in the above equations, we obtain

$$a = 0.20; \quad b = 0.15; \quad c = 0.05; \quad d = 0.60.$$

Then, the probabilities turn out to be

$$\mathbb{P}(B) = b + c = 0.20; \quad \mathbb{P}(A \cap B) = c = 0.05; \quad \mathbb{P}(A' \cup B) = b + c + d = 0.80,$$

and the conditional probabilities become

$$\mathbb{P}(B|A') = \frac{\mathbb{P}(B \cap A')}{\mathbb{P}(A')} = \frac{b}{b + d} = 0.20;$$
$$\mathbb{P}(A \cap B|A \cup B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} = \frac{c}{a + b + c} = 0.125.$$

□

Problem 2. *If events A, B, C are independent, show that*

(a) (7 pts) A and $B \cap C$ are independent.

(b) (13 pts) A and $B \cup C$ are independent.

Solution. (a) Since the events A, B, C are independent, they satisfy

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \quad \text{and} \quad \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C).$$

When we use the second equality in the first, we obtain

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(B)\mathbb{P}(A \cap C).$$

(b) We have

$$\begin{aligned}\mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}((A \cap B) \cup (A \cap C)) \\ &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cap (A \cap C)) \\ &= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B \cap C).\end{aligned}$$

Then we use the independence to continue as

$$\begin{aligned}\mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\ &= \mathbb{P}(A)[\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B)\mathbb{P}(C)] \\ &= \mathbb{P}(A)[\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)] \\ &= \mathbb{P}(A)\mathbb{P}(B \cup C),\end{aligned}$$

which gives the independence of A and $B \cup C$.

□

Problem 3. *We roll two fair dice. Let X be the random variable that gives the absolute value of the differences between the two numbers appearing on (upper) faces.*

(a) (4 pts) Find the range of X.

(b) (6 pts) Find the discrete probability function $f(x)$ of X.

(c) (10 pts) Find and plot the cumulative distribution function $F(x)$ of X.

Solution.

X	**1**	**2**	**3**	**4**	**5**	**6**
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Range $X = \{0, 1, 2, 3, 4, 5\}$. Then, it follows that

$$f(x) = \begin{cases} 6/36 & \text{if } x = 0, \\ 10/36 & \text{if } x = 1, \\ 8/36 & \text{if } x = 2, \\ 6/36 & \text{if } x = 3, \\ 4/36 & \text{if } x = 4, \\ 2/36 & \text{if } x = 5. \end{cases} \qquad F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 6/36 & \text{if } 0 \leq x < 1, \\ 16/36 & \text{if } 1 \leq x < 2, \\ 24/36 & \text{if } 2 \leq x < 3, \\ 30/36 & \text{if } 3 \leq x < 4, \\ 34/36 & \text{if } 4 \leq x < 5, \\ 1 & \text{if } 5 \leq x. \end{cases}$$

□

Problem 4. *A gambler has in his pocket a fair coin and a two-headed coin. He selects one of the coins at random, and when he flips it,*

(a) (6 pts) What is the probability that it comes up heads?

(b) (6 pts) If it comes up heads, then what's the probability that it's the fair coin?

(c) *(8 pts) Suppose that he flips the coin n times, and it comes up heads each time. What's the probability that it's fair?*

Solution. Let F be the event that the coin is fair and let H be the event that the toss results in heads.

(a) Obviously, we have $\mathbb{P}(F) = \mathbb{P}(F') = 0.5$. Then, by the law of total probability

$$\mathbb{P}(H) = \mathbb{P}(H|F)\mathbb{P}(F) + \mathbb{P}(H|F')\mathbb{P}(F') = (0.5)(0.5) + 1(0.5) = 0.75$$

(b) We use Bayes' theorem to find

$$\mathbb{P}(F|H) = \frac{\mathbb{P}(H|F)\mathbb{P}(F)}{\mathbb{P}(H|F)\mathbb{P}(F) + \mathbb{P}(H|F')\mathbb{P}(F')} = \frac{(0.5)(0.5)}{(0.5)(0.5) + 1(0.5)} = 1/3.$$

(c) Let H_n be the event that all of the n tosses result in heads. Then,

$$\mathbb{P}(F|H_n) = \frac{\mathbb{P}(H_n|F)\mathbb{P}(F)}{\mathbb{P}(H_n|F)\mathbb{P}(F) + \mathbb{P}(H_n|F')\mathbb{P}(F')} = \frac{(0.5)^n(0.5)}{(0.5)^n(0.5) + 1(0.5)} = 1/(2^n + 1).$$

□

Problem 5. *(10 pts) The probability of winning on a single toss of the dice is p. Player A starts, and if he fails, he passes the dice to B, who then attempts to win on her toss. They continue tossing back and forth until one of them wins. What are their probabilities of winning?*

Solution. Notice that A can win only in an odd repetition of the game (i.e., the first, third, fifth, and so on, repetition). Then we have

$$\mathbb{P}\{A \text{ wins}\} = \mathbb{P}\left(\bigcup_{j=0}^{\infty}\{A \text{ wins in the } (2j+1)\text{st toss}\}\right) = \sum_{j=0}^{\infty}\mathbb{P}\{A \text{ wins in the } (2j+1)\text{st toss}\}.$$

Notice now that if A wins in the $(2j+1)$st toss, for some $j = 0, 1, \ldots$, it is equivalent to A losing the 1st toss, B losing the 2nd toss, …, A losing the $(2j-1)$st toss, B losing the $(2j)$th toss, and A winning the $(2j+1)$st toss. Then,

$$\mathbb{P}\{A \text{ wins}\} = p\sum_{j=0}^{\infty}(1-p)^{2j} = \frac{1}{2-p}.$$

For the probability that player B wins the game we simply have

$$\mathbb{P}\{B \text{ wins}\} = 1 - \mathbb{P}\{A \text{ wins}\} = \frac{1-p}{2-p}.$$

□

Second Solution. If A loses the first toss, and B loses the second one the game starts afresh (as if the first two tosses never happened). Then the probability that A wins is simply

$$\mathbb{P}\{A \text{ wins}\} = p + (1-p)^2\mathbb{P}\{A \text{ wins}\},$$

which implies that

$$\mathbb{P}\{A \text{ wins}\} = \frac{p}{1-(1-p)^2} = \frac{1}{2-p}.$$

Similarly,

$$\mathbb{P}\{B \text{ wins}\} = (1-p)p + (1-p)^2\mathbb{P}\{B \text{ wins}\},$$

which yields

$$\mathbb{P}\{P \text{ wins}\} = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}.$$

□

Problem 6. *(10 pts) Verify that the following functions, with $0 < p < 1$, can serve as the probability distribution of a random variable X within the given range;*

(a) $f(x) = \mathbb{P}(X = x) = p(1-p)^{x-1}$, *for* $x = 1, 2, \ldots$.

(b) $f(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$, *for* $x = 0, 1, \ldots, n$.

Note that these functions are known as the ***geometric distribution*** *with parameter p and the* ***binomial distribution*** *with paramaters n and p, respectively.*

Solution. Obviously, $f(x) \geq 0$ in both cases.

(a) Using the geometric series formula,

$$\sum_{x=1}^{\infty} p(1-p)^{x-1} = \frac{p}{1-(1-p)} = 1.$$

(b) Using the binomial formula,

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = [p + (1-p)]^n = 1.$$

□

▶ The following result is known as **Vandermonde's identity**:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}, \quad m, n, r \in \mathbb{N}.$$

Problem 7. *(Bonus 10 pts) Prove Vandermonde's identity, using a combinatorial argument.*
(Hint: Consider the ways of selecting r people from a group of m men and n women.)

Solution. We consider the ways of selecting r people from a group of m men and n women.

LHS: Among a group of $m + n$ people, r people can be selected in $\binom{m+n}{r}$ many ways.

RHS: To select r people from a group of m men and n women, one can select no man and r women, or select 1 man and $r - 1$ women, or ..., select r men and no woman, namely, there are

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

many ways to do so.

□

Problem 8. *(Bonus 10 pts) Prove Vandermonde's identity, using the binomial theorem.*
(Hint: Consider the coefficient of y^r in $(1+y)^{m+n}$.)

Solution. We consider the coefficient of y^r in $(1+y)^{m+n}$.

LHS: The coefficient of y^r in $(1+y)^{m+n}$ is $\binom{m+n}{r}$.

RHS: We write $(1+y)^{m+n} = (1+y)^m(1+y)^n$. The coefficient of y^r in the expansion

$$(1+y)^m(1+y)^n = \left[\binom{m}{0} + \binom{m}{1} y + \cdots + \binom{m}{m} y^m\right]\left[\binom{n}{0} + \binom{n}{1} y + \cdots + \binom{n}{n} y^n\right]$$

is given by

$$\binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

□