MIDTERM EXAM II

Problem 1. *The cumulative distribution function of the continuous random variable X is given by*

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ ax^2 & \text{if } 0 \le x < 1, \\ 1/2 & \text{if } 1 \le x < 2, \\ bx - c & \text{if } 2 \le x < 3, \\ 1 & \text{if } 3 \le x. \end{cases}$$

(a) (8 pts) Find the values of the constants a, b, c and sketch F.

(b) (6 pts) Find the probability density function $f(x)$ of X.

(c) (6 pts) Calculate $\mathbb{P}(|X-1| < 1/2)$.

Solution. (a) By continuity of F, it follows that

$$\lim_{x\to 1^-} F(x) = F(1) \Rightarrow a = \frac{1}{2}; \quad \lim_{x\to 2^-} F(x) = F(2) \Rightarrow \frac{1}{2} = 2b - c; \quad \lim_{x\to 3^-} F(x) = F(3) \Rightarrow 1 = 3b - c.$$

So, we obtain $b = c = 1/2$.

(b) By definition,

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ 1/2 & \text{if } 2 \le x < 3, \\ 0 & \text{elsewhere.} \end{cases}$$

(c) We compute

$$\mathbb{P}(|X-1| < 1/2) = \mathbb{P}(1/2 < X < 3/2) = F(3/2) - F(1/2) = 3/8.$$

□

Problem 2. *The probability density of X is given by $f(x) = \frac{1}{2}e^{-|x-1|}$, $x \in \mathbb{R}$.*

(a) (10 pts) Find the moment-generating function of X.

(b) (10 pts) Use the moment-generating function to find the expected value and variance of X.

We are given that

Solution. (a)

$$f(x) = \begin{cases} \frac{1}{2}e^{x-1} & \text{if } x \le 1, \\ \frac{1}{2}e^{1-x} & \text{if } x > 1. \end{cases}$$

Then,

$$M_X(t) = \mathbb{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{1} e^{tx}\frac{1}{2}e^{x-1}\, dx + \int_{1}^{\infty} e^{tx}\frac{1}{2}e^{1-x}\, dx = \frac{e^t}{1-t^2}.$$

(b) We compute

$$\mathbb{E}(X) = \frac{d}{dt}M_X(t)\bigg|_{t=0} = \frac{e^t(1-t^2)+2te^t}{(1-t^2)^2}\bigg|_{t=0} = 1; \quad \mathbb{E}(X^2) = \frac{d^2}{dt^2}M_X(t)\bigg|_{t=0} = 3.$$

Therefore,

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = 3 - 1 = 2$$

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□

Problem 3. *Let X be a geometric distributed random variable with success probability $0 < p < 1$.*

(a) (6 pts) Compute $\mathbb{P}(X \text{ is odd})$.

(b) (6 pts) Show that $\mathbb{P}(X > n) = (1-p)^n$, when n is a positive integer.

(c) (8 pts) Justify the memoryless property of the geometric distribution, that is, for any x,

$$\mathbb{P}(X > k + x \mid X > k) = \mathbb{P}(X > x), \quad k = 1, 2, \dots.$$

Solution. X being a geometric random variable, its probability distribution is given by

$$f(x) = \mathbb{P}(X = x) = p(1-p)^{x-1}, \quad x = 1, 2, \dots.$$

(a) We compute

$$\mathbb{P}(X \text{ is odd}) = \sum_{x=1}^{\infty} f(2x-1) = \sum_{x=1}^{\infty} p(1-p)^{2x-2} = \frac{1}{2-p}.$$

(b) We compute

$$\mathbb{P}(X > n) = \sum_{x=n+1}^{\infty} p(1-p)^{x-1} = (1-p)^n.$$

(c) We compute

$$\mathbb{P}(X > k + x \mid X > k) = \frac{\mathbb{P}(X > k + x)}{\mathbb{P}(X > x)} = \frac{(1-p)^{k+x}}{(1-p)^k} = (1-p)^x = \mathbb{P}(X > x).$$

□

Problem 4. *A lottery has a 1/400 chance of winning a prize. You play the lottery 200 times.*

(a) (6 pts) What is the probability that you win at least twice? Give an exact answer.

(b) (6 pts) Find the probability that you win at least twice, but now using a suitable approximation.

(c) (8 pts) Suppose you know that you have won at least once. What is the probability that you have won at least twice? Use a suitable approximation to answer the question.

Solution. (a) Let the random variable X be the number of successes in 200 lottery games. Then, X is a binomial random variable with parameters $n = 200$ and $p = 1/400$. We compute

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) = 1 - \binom{200}{0}\left(\frac{399}{400}\right)^{200} - \binom{200}{1}\left(\frac{1}{400}\right)^{1}\left(\frac{399}{400}\right)^{199}.$$

(b) Since $n = 200$ is large and $p = 1/400$ is small, Poisson approximation is ideal for this situation. We have $\lambda = np = 1/2$, so Poisson approximation gives for the above probability

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X = 0) - \mathbb{P}(X = 1) \approx 1 - \frac{e^{-1/2}(1/2)^0}{0!} - \frac{e^{-1/2}(1/2)^1}{1!} = 1 - \frac{3}{2}e^{-1/2}.$$

(c) The probability sought is the conditional probability $\mathbb{P}(X \geq 2 | X \geq 1)$. Using again Poisson approximation, the probability asked is

$$\mathbb{P}(X \geq 2 | X \geq 1) = \frac{\mathbb{P}(X \geq 2)}{\mathbb{P}(X \geq 1)} = \frac{1 - \frac{3}{2}e^{-1/2}}{1 - e^{-1/2}}.$$

□

Problem 5. *(10 pts) My friend and I play the following game: I pay her \$2 to play. I flip a coin at most 5 times. If it comes up heads the first time, I get a \$1 and the game ends, otherwise, I flip again and if it comes up heads this time, I get \$2. We continue in this way, so that if the first heads is on the third toss I get \$4, the fourth toss \$8, and the fifth toss \$16. What is my expectation for this game?*

Solution. Let X denotes the amount of money (in dollars) that she pays me back. Then, $Range(X) = \{1, 2, 4, 8, 16\}$ with the following probabilities $f(x) = \mathbb{P}(X = x)$;

$$f(1) = 1/2; \quad f(2) = 1/4; \quad f(4) = 1/8; \quad f(8) = 1/16; \quad f(16) = 1/32.$$

Thus, the expected value that she pays me back is

$$\sum_x x f(x) = 1f(1) + 2f(2) + 4f(3) + 8f(4) + 16f(5) = 2.5.$$

Therefore, my expectation is \$2.5 − \$2 = \$0.5. □

Problem 6. *Sergio Rodriguez, who plays for Real Madrid Baloncesto (basketball club), has 90 % free throw percentage in Euroleague. During the season,*

(a) (4 pts) What is the probability that he makes his first free throw on his fifth shot?

(b) (6 pts) What is the probability that he makes his third free throw on his fifth or sixth shot?

Solution. (a) Geometric distribution;

$$\mathbb{P}(X = 5) = (0.1)^4(0.9).$$

(b) Negative binomial distribution;

$$\mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{5-1}{3-1}(0.9)^3(0.1)^2 + \binom{6-1}{3-1}(0.9)^3(0.1)^3.$$

□

Problem 7. *(Bonus 10 pts) Given that the random variable X has moment-generating function*

$$M_X(t) = \frac{1}{4}e^{-5t} + \frac{1}{8}e^{-3t} + \frac{1}{6}e^{-t} + \frac{1}{12}e^{2t} + \frac{3}{8}e^{3t},$$

compute the probabilities $\mathbb{P}(-4 < X \leq 2)$ and $\mathbb{P}(X = 3)$.

Solution. Comparing the given formula $M_X(t)$ with the general formula for a m.g.f. of a discrete distribution, we see that X must have values $\{-5, -3, -1, 2, 3\}$ with probabilities $\{1/4, 1/8, 1/6, 1/12, 3/8\}$, respectively. Therefore,

$$\mathbb{P}(-4 < X \leq 2) = f(-3) + f(-1) + f(2) = 3/8; \qquad \mathbb{P}(X = 3) = f(3) = 3/8.$$

□

Problem 8. *(Bonus 10 pts) Prove the non-negativity of the* ***variance****, using Jensen's inequality.* **(Hint: Consider the convex function $g(x) = x^2$.)**

Solution. Jensen's inequality states that if g is a convex function, then $\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$. Hence, taking $g(x) = x^2$ implies $\mathbb{E}(X^2) \geq \mathbb{E}^2(X)$. Therefore, $Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) \geq 0$. □