Recitation 3

1. Suppose you pick a card at random from a well shuffled deck. What is the probability that it is an Ace given that it is a Diamond?

   Solution: Here, the sample space Ω has 52 equally likely outcomes. Let A be the event that the card is an Ace, and D be the event that it is a Diamond. Then we compute

$$\mathbb{P}(A|D) = \frac{\mathbb{P}(A \cap D)}{\mathbb{P}(D)} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}.$$

2. Exercise 2.95: A shipment of 1000 parts contains 1 percent defective parts. Find the probability that

   (a) the first four parts chosen arbitrarily for inspection are nondefective;

   (b) the first defective part found will be on the fourth inspection.

   Solution:

   (a) In set notation, let A_i be the event that the ith item is non-defective. Then we are asked to compute $\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$. Note that we can write

$$\begin{aligned}\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) &= \frac{\mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)}{\mathbb{P}(A_1 \cap A_2 \cap A_3)} \cdot \frac{\mathbb{P}(A_1 \cap A_2 \cap A_3)}{\mathbb{P}(A_1 \cap A_2)} \cdot \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_1)} \cdot \mathbb{P}(A_1) \\ &= \mathbb{P}(A_4|A_1 \cap A_2 \cap A_3) \cdot \mathbb{P}(A_3|A_1 \cap A_2) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_1).\end{aligned}$$

   In plain words, this is equal to (going backwards)

   the probability that first item is non-defective $\times$
   the probability that second item is non-defective given that so is the first $\times$
   the probability that third item is non-defective given that so are the first two $\times$
   the probability that fourth item is non-defective given that so are the first three .

   Clearly, the probability that first item is non-defective equals 990/1000, the probability that second item is non-defective given that so is the first is equal to 989/999, the probability that third item is non-defective given that so are the first two is 988/998, and finally the probability that fourth item is non-defective given that so are the first three is 987/997. Substituting these values we get

$$\frac{990}{1000} \cdot \frac{989}{999} \cdot \frac{988}{998} \cdot \frac{987}{997} \approx 0.9605.$$

   Note for TA's: The question is of course easy to solve, but it is also helpful to make an explanation as above to help students understand the math behind it in terms of conditional probabilities.

   (b) This time, the first three item will be nondefective and fourth will be defective. That is, we compute

$$\frac{990}{1000} \cdot \frac{989}{999} \cdot \frac{988}{998} \cdot \frac{1}{10} \approx 0.0970$$

3. Exercise 2.98: Medical records show that one out of 10 persons in a certain town has a thyroid deficiency. If 12 persons in this town are randomly chosen and tested, what is the probability that at least one of them will have a thyroid deficiency?

   Solution: Let A_i be the event that ith selected person is healthy / does not have the thyroid deficiency. Then the probability that we compute is

$$1 - \mathbb{P}(A_1 \cap \ldots \cap A_{12}).$$

   Here we assume that the events A_is are independent. Hence, we have

$$1 - \mathbb{P}(A_1 \cap \ldots \cap A_{12}) = 1 - \mathbb{P}(A_1) \ldots \mathbb{P}(A_{12}) = 1 - (0.9)^{12} \approx 0.7176$$

4. Exercise 2.103: It is known from experience that in a certain industry 60 percent of all labor-management disputed are over wages, 15 percent are over working conditions, and 25 percent are over fringe issues. Also, 45 percent of the disputes over wages are resolved without strikes, 70 percent of the disputed over working conditions are resolved without strikes, and 40 percent of the disputes over fringes are resolved without strikes. What is the probability that a labor-management dispute in this industry will be resolved without a strike?

   Solution: Let W be the event that the dispute is over wages, C be the event that it is over working conditions, and F be the event that it is over fringe issues. Also, S be the event that there is no strike. The we compute

$$\begin{aligned}\mathbb{P}(S) &= \mathbb{P}(S \cap W) + \mathbb{P}(S \cap C) + \mathbb{P}(S \cap F)\\ &= \mathbb{P}(S|W) \cdot \mathbb{P}(W) + \mathbb{P}(S|C) \cdot \mathbb{P}(C) + \mathbb{P}(S|F) \cdot \mathbb{P}(F)\\ &= 0.45 \cdot 0.6 + 0.7 \cdot 0.15 + 0.4 \cdot 0.25 = 0.475.\end{aligned}$$

5. You pick two cards one after another from a well-shuffled deck of 52 cards. Given that you picked one Ace and one King, what is the probability that the first card drawn was Ace?

   Solution: Let A be the event that the first card is an Ace and let B be the event that we pick one Ace and one King. Note that we have

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A')\\ &= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A') \cdot \mathbb{P}(A')\\ &= \frac{4}{52} \cdot \frac{4}{51} + \frac{4}{52} \cdot \frac{4}{51}.\end{aligned}$$

   (Note that $B \cap A$ is the event that we fist pick an Ace then a King, and $B \cap A'$ is the event that we first pick a King and then an Ace.)

   Then we compute

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)} = \frac{\frac{4}{52} \cdot \frac{4}{51}}{\frac{4}{52} \cdot \frac{4}{51} + \frac{4}{52} \cdot \frac{4}{51}} = \frac{1}{2}.$$

6. An urn contains two white balls and one black ball, whereas a second urn has one white and five black balls. A ball is drawn at random from the first urn and without looking at its color it is placed in the urn 2. Then a ball is drawn at random from urn 2.

(a) What is the probability that the final ball is white?

(b) If the ball drawn from urn 2 is white, what is the probability that the transferred ball was black? / was white?

Solution: For $i = 1, 2$, let W_i be the event that the ball picked in the ith draw is white (note that W_i' be the event that the ball in the ith draw is black).

(a) We have

$$\begin{aligned}\mathbb{P}(W_2) &= \mathbb{P}(W_2 \cap W_1) + \mathbb{P}(W_2 \cap W_1') \\ &= \mathbb{P}(W_2|W_1) \cdot \mathbb{P}(W_1) + \mathbb{P}(W_2|W_1') \cdot \mathbb{P}(W_1') \\ &= \frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3} = \frac{5}{21}\end{aligned}$$

(b) The probability that the transferred ball was black given that the ball drawn from urn 2 is white can be computed as

$$\begin{aligned}\mathbb{P}(W_1'|W_2) &= \frac{\mathbb{P}(W_1' \cap W_2)}{\mathbb{P}(W_2)} \\ &= \frac{\mathbb{P}(W_2|W_1') \cdot \mathbb{P}(W_1')}{\mathbb{P}(W_2)} \\ &= \frac{\frac{1}{7} \cdot \frac{1}{3}}{\frac{5}{21}} = \frac{1}{5}.\end{aligned}$$

Using this result, we cam compute the conditional probability that the transferred ball was white given that the ball drawn from urn 2 is white as

$$\mathbb{P}(W_1|W_2) = 1 - \mathbb{P}(W_1'|W_2) = \frac{4}{5}.$$

7. A blood test is 95% effective in detecting a disease when it is in fact present. However, the test also yields a false positive for 0.1% of healthy persons tested. If 0.5% of the population has the disease, what is the probability that a person has the disease given that the test result is positive?

Solution: Let D be the event that the person tested has the disease and let P be the event that the test yields a "positive" result. The question states that

$$\mathbb{P}(P|D) = 0.95, \qquad \mathbb{P}(P|D') = 0.001, \qquad \mathbb{P}(D) = 0.005.$$

Using these probabilities we compute

$$\begin{aligned}\mathbb{P}(P) &= \mathbb{P}(P \cap D) + \mathbb{P}(P \cap D') \\ &= \mathbb{P}(P|D) \cdot \mathbb{P}(D) + \mathbb{P}(P|D') \cdot \mathbb{P}(D') \\ &= 0.95 \cdot 0.005 + 0.001 \cdot 0.995 = 0.005745,\end{aligned}$$

in terms of which we have

$$\mathbb{P}(D|P) = \frac{\mathbb{P}(D \cap P)}{\mathbb{P}(P)} = \frac{\mathbb{P}(P|D) \cdot \mathbb{P}(D)}{\mathbb{P}(B)} = \frac{0.95 \cdot 0.005}{0.005745} = 0.826806.$$

Alternatively, we can directly write

$$\mathbb{P}(D|P) = \frac{\mathbb{P}(D \cap P)}{\mathbb{P}(P)} = \frac{\mathbb{P}(P|D) \cdot \mathbb{P}(D)}{\mathbb{P}(P|D) \cdot \mathbb{P}(D) + \mathbb{P}(P|D') \cdot \mathbb{P}(D')} = \frac{0.95 \cdot 0.005}{0.95 \cdot 0.005 + 0.001 \cdot 0.995},$$

which gives the same answer.

8. In answering a question with four choices in a multiple choice exam, a student either knows the answer of guesses it. Let $p \in (0, 1)$ be the probability that the student knows the answer. A student guessing the answer will select the correct answer with probability 1/4.

   What is the probability that the student knew the answer given that s/he answered correctly?

   Solution: Let C be the event that the answer is correct and let K be the event that the students knows the answer/subject.

   The question states that

$$\mathbb{P}(K) = p, \qquad \mathbb{P}(C|K') = 1/4, \qquad \text{(and obviously } \mathbb{P}(C|K) = 1).$$

   Then, we can directly compute

$$\mathbb{P}(K|C) = \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C|K) \cdot \mathbb{P}(K)}{\mathbb{P}(C|K) \cdot \mathbb{P}(K) + \mathbb{P}(C|K') \cdot \mathbb{P}(K')} = \frac{1 \cdot p}{1 \cdot p + \frac{1}{4} \cdot (1 - p)} = \frac{4p}{1 + 3p}$$

9. One coin amongst two in an urn is double headed, the rest are fair. Your friends selects one of these coins at random and tosses it n times, recording Heads every time. What is the probability that the selected coin is a fair one?

   Solution: Let F be the event that the coin is fair and let A_n be the event that all tosses result in Heads.

   Then, we compute

$$\mathbb{P}(F|A_n) = \frac{\mathbb{P}(A_n \cap F)}{\mathbb{P}(A_n)} = \frac{\mathbb{P}(A_n|F) \cdot \mathbb{P}(F)}{\mathbb{P}(A_n|F) \cdot \mathbb{P}(F) + \mathbb{P}(A_n|F') \cdot \mathbb{P}(F')} = \frac{2^{-n} \cdot \frac{1}{2}}{2^{-n} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}} = \frac{1}{1 + 2^n}.$$

   Note that $\mathbb{P}(F|A_n)$ goes to 0 as $n \to \infty$.

10. Exercise 2.28: If $\mathbb{P}(A|B) < \mathbb{P}(A)$, prove that $\mathbb{P}(B|A) < \mathbb{P}(B)$.

    Solution: From the statement of the question, it follows that we have $\mathbb{P}(B) > 0$; otherwise $\mathbb{P}(A|B)$ would not be well-defined. Also, since $\mathbb{P}(A|B) \geq 0$, $\mathbb{P}(A) > 0$ as well.

    The statement $\mathbb{P}(A|B) < \mathbb{P}(A)$ can be re-written as

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} < \mathbb{P}(A).$$

    Multiplying both sides with $\mathbb{P}(B)/\mathbb{P}(A) > 0$ we obtain

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} < \mathbb{P}(B),$$

    and this gives the desired result since $\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \mathbb{P}(B|A)$.