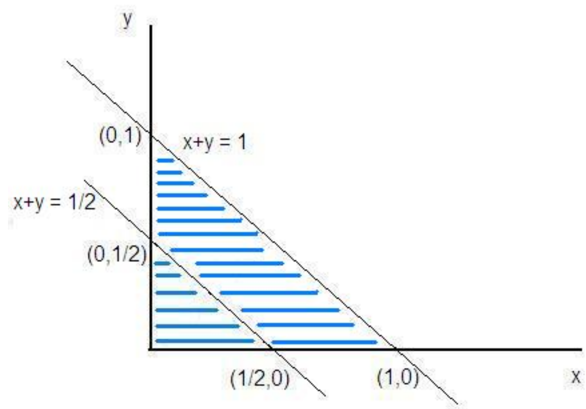
Recitation 4

1. Exercise 3.50: If the joint probability density of X and Y is given by

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x < 1,\ 0 < y < 1,\ x+y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find $\mathbb{P}(X + Y < \frac{1}{2})$.

Solution:



The blue shaded region is the set of points (x, y)'s for which $f(x, y) > 0$ (the joint probability density function is zero everywhere else), and the probability that we need to find is the integral of $f(x, y)$ over the portion of the blue region staying below the line $x+y = \frac{1}{2}$. Carrying out this integration, we obtain

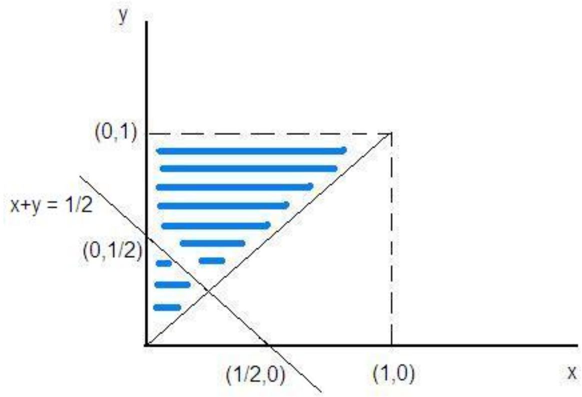
$$\begin{aligned} \int_0^{1/2} \int_0^{\frac{1}{2}-x} 24xy\,dy\,dx &= \int_0^{1/2} 12x \left(\frac{1}{2} - x\right)^2 dx \\ &= \int_0^{1/2} 12x \left(\frac{1}{4} + x^2 - x\right) dx \\ &= \int_0^{1/2} \left(3x + 12x^3 - 12x^2\right) dx = \frac{3}{8} + \frac{3}{16} - \frac{1}{2} = \frac{1}{16}. \end{aligned}$$

2. Exercise 3.53: If the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \dfrac{1}{y} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

find the probability that the sum of the values of X and Y will exceed 1/2.

Solution:



The blue shaded region is the set of points (x, y)'s for which $f(x, y) > 0$ (the joint probability density function is zero everywhere else). Note that the probability that we need to find is $\mathbb{P}(X + Y > 1/2)$, and this is given by the integral of $f(x, y)$ over the portion of the blue region staying above the line $x + y = \frac{1}{2}$.

Alternatively, we can compute the integral of $f(x, y)$ over the lower portion of the blue region and subtract the value of the integral from one. That is, we compute

$$1 - \left(\int_0^{1/4} \int_0^y \frac{1}{y}\, dx\, dy + \int_{1/4}^{1/2} \int_0^{\frac{1}{2}-y} \frac{1}{y}\, dx\, dy \right)$$

$$= 1 - \left(\int_0^{1/4} 1\, dy + \int_{1/4}^{1/2} \left(\frac{1}{2y} - 1 \right) dy \right) = 1 - \left(\frac{1}{4} + \frac{1}{2}(\ln(1/2) - \ln(1/4)) - \frac{1}{4} \right) = 1 - \frac{1}{2} \ln 2.$$

3. Exercise 3.54: Find the joint probability density of the two random variables X and Y whose joint distribution is given by

$$F(x, y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}) & \text{for } x > 0,\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Solution: We have

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$$

wherever the partial derivatives exist. Partial differentiation yields

$$\frac{\partial^2}{\partial x \partial y} F(x, y) = 4xy\, e^{-(x^2+y^2)}$$

for $x > 0$ and $y > 0$ and 0 elsewhere. So we have

$$f(x, y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & \text{for } x > 0,\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

4. Exercise 3.62: Find k if the joint the probability distribution of X, Y, and Z is given by

$$f(x, y, z) = kxyz$$

for $x = 1, 2$; $y = 1, 2, 3$; $z = 1, 2$.

Solution: Clearly, k must be non-negative. To find its value, we use the condition that the sum of $f(x, y, z)$ values for $x = 1, 2$ $y = 1, 2, 3$ and $z = 1, 2$ is equal to one. That is, we must have $\sum_{x=1}^{2} \sum_{y=1}^{3} \sum_{z=1}^{2} f(x, y, z) = 1$. Note that the sum can be computed as

$$\sum_{x=1}^{2} \sum_{y=1}^{3} \sum_{z=1}^{2} f(x, y, z) = \sum_{x=1}^{2} \sum_{y=1}^{3} \sum_{z=1}^{2} kxyz = \sum_{x=1}^{2} \sum_{y=1}^{3} 3kxy = \sum_{x=1}^{2} 18kx = 54k.$$

Hence $k = 1/54$.

5. Exercise 3.63: With reference to Exercise 62, find

   (a) $\mathbb{P}(X = 1, Y \leq 2, Z = 1)$.

   (b) $\mathbb{P}(X = 2, Y + Z = 4)$.

   Solution:

   (a) $\mathbb{P}(X = 1, Y \leq 2, Z = 1) = f(1, 1, 1) + f(1, 2, 1) = 3/54 = 1/18$.

   (b) $\mathbb{P}(X = 2, Y + Z = 4) = f(2, 2, 2) + f(2, 3, 1) = 14/54 = 7/27$.

6. Exercise 3.68: If the joint probability density of X, Y, and Z is given by

$$f(x, y, z) = \begin{cases} \frac{1}{3}(2x + 3y + z) & \text{for } 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   find

   (a) $\mathbb{P}(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2})$.

   (b) $\mathbb{P}(X < \frac{1}{2}, Y < \frac{1}{2}, Z < \frac{1}{2})$.

   Solution:

   (a) We have

$$\mathbb{P}\left(X = \frac{1}{2}, Y = \frac{1}{2}, Z = \frac{1}{2}\right) = \int_1^1 \int_1^1 \int_1^1 f(x, y, z)\, dz\, dz\, dz = 0.$$

   (b) We compute

$$\begin{aligned} \mathbb{P}\left(X < \frac{1}{2}, Y < \frac{1}{2}, Z < \frac{1}{2}\right) &= \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}} \int_{-\infty}^{\frac{1}{2}} f(x, y, z)\, dz\, dy\, dx \\ &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{1}{3}(2x + 3y + z)\, dz\, dy\, dx \\ &= \frac{1}{3} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} (2x + 3y + z)\, dz\, dy\, dx \\ &= \frac{1}{3} \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left(x + \frac{3}{2}y + \frac{1}{8}\right) dy\, dx \\ &= \frac{1}{3} \int_0^{\frac{1}{2}} \left(\frac{1}{2}x + \frac{1}{4}\right) dx = \frac{1}{3}\left(\frac{1}{16} + \frac{1}{8}\right) = \frac{1}{16}. \end{aligned}$$

7. Exercise 70 (a-b): With reference to Exercise 42 (see page 90); find

   (a) the marginal distribution of X;

   (b) the marginal distribution of Y.

   Solution:

   (a) Let $g(x)$ be the marginal distribution of X. Then, for $x = 0, 1, 2$, we compute

$$\begin{aligned} g(0) &= \frac{1}{12} + \frac{1}{4} + \frac{1}{8} + \frac{1}{120} = \frac{56}{120} = \frac{7}{15} \\ g(1) &= \frac{1}{6} + \frac{1}{4} + \frac{1}{20} = \frac{28}{60} = \frac{7}{15} \\ g(2) &= \frac{1}{24} + \frac{1}{40} = \frac{8}{120} = \frac{1}{15} \end{aligned}$$

   (b) Let $h(y)$ be the marginal distribution of Y. Then, for $y = 0, 1, 2, 3$, we compute

$$\begin{aligned} h(0) &= \frac{1}{12} + \frac{1}{6} + \frac{1}{24} = \frac{7}{24} \\ h(1) &= \frac{1}{4} + \frac{1}{4} + \frac{1}{40} = \frac{21}{40} \\ h(2) &= \frac{1}{8} + \frac{1}{20} = \frac{7}{40} \\ h(3) &= \frac{1}{120} \end{aligned}$$

8. Exercise 71 (a-c): Given the joint probability distribution

$$f(x, y, z) = \frac{xyz}{108}, \qquad \text{for } x = 1, 2, 3;\ y = 1, 2, 3;\ z = 1, 2$$

   find

   (a) the joint marginal distribution of X and Y;

   (b) the joint marginal distribution of X and Z;

   (c) the marginal distribution of X.

   Solution:

   (a) Let $g(x, y)$ be the joint marginal distribution of X and Y. Then,

$$g(x, y) = \sum_{z=1}^{2} f(x, y, z) = \sum_{z=1}^{2} \frac{xyz}{108} = \frac{xy}{36}, \qquad \text{for } x = 1, 2, 3;\ y = 1, 2, 3.$$

   (b) Let $h(x, z)$ be the joint marginal distribution of X and Z. Then, we compute it as

$$h(x, z) = \sum_{y=1}^{3} f(x, y, z) = \sum_{y=1}^{3} \frac{xyz}{108} = \frac{xz}{18}, \qquad \text{for } x = 1, 2, 3;\ z = 1, 2.$$

   (c) Let $\ell(x)$ the marginal distribution of X. We can compute it as

$$\ell(x) = \sum_{y=1}^{3} \sum_{z=1}^{2} f(x, y, z) = \sum_{y=1}^{3} \sum_{z=1}^{2} \frac{xyz}{108} = \sum_{y=1}^{3} \frac{xy}{36} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3,$$

Equivalently,

$$\ell(x) = \sum_{y=1}^{3} g(x,y) = \sum_{y=1}^{3} \frac{xy}{36} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3.$$

or

$$\ell(x) = \sum_{z=1}^{2} h(x,z) = \sum_{z=1}^{2} \frac{xz}{18} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3.$$

9. Exercises 74 a & 75 a: If the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{4}(2x+y) & \text{for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find

(a) the marginal density of X;

(b) the marginal density of Y.

Solution:

(a) Let $g(x)$ be the marginal density of X. Clearly, for $x \notin (0,1)$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 0.$$

For $x \in (0,1)$, we have

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{2} \frac{1}{4}(2x+y)\,dy = \frac{1}{4}(4x+2) = \frac{1}{2}(2x+1).$$

Hence,

$$g(x) = \begin{cases} \frac{1}{2}(2x+1) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) Let $h(y)$ be the marginal density of Y. Likewise, for $y \notin (0,2)$,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = 0.$$

For $y \in (0,2)$, we have

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{1} \frac{1}{4}(2x+y)\,dx = \frac{1}{4}(1+y).$$

Hence,

$$h(y) = \begin{cases} \frac{1}{4}(1+y) & \text{for } 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

10. Exercise 77: With reference to exercise 53, find

(a) the marginal density of X;

(b) the marginal density of Y.

Solution:

(a) Let $g(x)$ be the marginal density of X. Clearly, for $x \notin (0,1)$,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 0.$$

For $x \in (0,1)$, we have

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{x}^{1} \frac{1}{y}\,dy = \ln y\big|_x^1 = -\ln x.$$

(**Note for TA's:** mention that $\ln x < 0$, for $x \in (0,1)$.)

Hence, we have

$$g(x) = \begin{cases} -\ln x & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) Let $h(y)$ be the marginal density of Y. Likewise, for $y \notin (0,1)$,

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = 0.$$

For $y \in (0,1)$, we have

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{y} \frac{1}{y}\,dx = 1.$$

Hence,

$$h(y) = \begin{cases} 1 & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Recitation 6

1. Let X and Y be two discrete random variables with the joint probability distribution

$$f(x,y) = \frac{1}{21}(x+y), \qquad \text{for } x = 1,2,3;\ y = 1,2.$$

Find

(a) the marginal distribution of X;

(b) the conditional distribution of Y given $X = 1$.

Solution:

(a) Let $f_X(x)$, for $x = 1,2,3$, be the marginal distribution of X. Then we have

$$f_X(x) = \sum_{y=1}^{2} f(x,y) = \sum_{y=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}(2x+3), \qquad \text{for } x = 1,2,3.$$

(b) To find the conditional distribution of Y given $X = 1$, we compute

$$f_{Y|X}(y|1) = \frac{f(1,y)}{f_X(1)}, \qquad \text{for } y = 1,2.$$

Note that $f_X(1)$ must be different than zero for the conditional distribution to make sense. In this case, we have $f_X(1) = \frac{5}{21} \neq 0$.
Carrying out the computation, we obtain

$$f_{Y|X}(y|1) = \frac{f(1,y)}{f_X(1)} = \frac{\frac{1}{21}(1+y)}{\frac{5}{21}} = \frac{1}{5}(1+y), \qquad \text{for } y = 1,2.$$

2. Let X and Y be two continuous random variables with the joint probability density

$$f(x,y) = \begin{cases} 24xy & \text{for } 0 < x < 1,\ 0 < y < 1,\ x+y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(a) the marginal density of Y;

(b) the conditional density of X given $Y = 1/2$.

Solution:

Note for TA's: draw the x-y axis and illustrate the region over which $f(x,y) > 0$

(a) Let $f_Y(x)$ be the marginal density of Y. Clearly, for $y \notin (0,1)$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = 0.$$

For $y \in (0,1)$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 24xy\,dx = 12y \int_0^{1-y} 2x\,dx = 12y(1-y)^2$$

Hence,

$$f_Y(y) = \begin{cases} 12y(1-y)^2 & \text{for } 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Note that in terms of the indicator function

$$1_{(0,1)}(y) = \begin{cases} 1 & \text{for } y \in (0,1) \\ 0 & \text{for } y \notin (0,1) \end{cases}$$

we can rewrite the marginal density of Y as

$$f_Y(y) = 12y(1-y)^2 \, 1_{(0,1)}(y), \qquad \text{for } -\infty < y < \infty.$$

(b) The function

$$f_{X|Y}\left(x \middle| \frac{1}{2}\right) = \frac{f(x, \frac{1}{2})}{f_y(\frac{1}{2})}, \qquad \text{for } -\infty < x < \infty$$

gives the conditional density of X given $Y = \frac{1}{2}$. Note that $f_Y(\frac{1}{2})$ must be different from zero for this definition to make sense. Here, $f_Y(\frac{1}{2}) = \frac{3}{2} \neq 0$.
For $x \notin (0, \frac{1}{2})$, $f(x, \frac{1}{2}) = 0$, and therefore $f_{X|Y}(x|\frac{1}{2}) = 0$.
For $x \in (0, \frac{1}{2})$, we have $f(x, \frac{1}{2}) = 12x$, and this gives

$$f(x\,|1) = \frac{12x}{\frac{3}{2}} = 8x.$$

Hence, we have

$$f_{X|Y}\left(x \middle| \frac{1}{2}\right) = \begin{cases} 8x & \text{for } 0 < x < \dfrac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Once again, in terms of the indicator function

$$1_{(0,\frac{1}{2})}(x) = \begin{cases} 1 & \text{for } x \in \left(0, \dfrac{1}{2}\right) \\ 0 & \text{for } x \notin \left(0, \dfrac{1}{2}\right) \end{cases}$$

we can rewrite the marginal density of Y as

$$f_{X|Y}\left(x \middle| \frac{1}{2}\right) = 8x \, 1_{(0,\frac{1}{2})}(x), \qquad \text{for } -\infty < x < \infty.$$

Reminder: For a set A, the indicator function $1_A(x)$ is defined as

$$1_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A. \end{cases}$$

3. Exercise 70 (c-d): With reference to Exercise 42 (see page 90); find

(c) the conditional distribution of X given $Y = 1$;

(d) the conditional distribution of Y given $X = 0$.

Solution: In Recitation 5, we solved parts (a) and (b) as follows:

(a) Let $f_X(x)$ be the marginal distribution of X. Then, for $x = 0, 1, 2$, we compute

$$f_X(0) = \frac{1}{12} + \frac{1}{4} + \frac{1}{8} + \frac{1}{120} = \frac{56}{120} = \frac{7}{15}$$
$$f_X(1) = \frac{1}{6} + \frac{1}{4} + \frac{1}{20} = \frac{28}{60} = \frac{7}{15}$$
$$f_X(2) = \frac{1}{24} + \frac{1}{40} = \frac{8}{120} = \frac{1}{15}$$

(b) Let $f_Y(y)$ be the marginal distribution of Y. Then, for $y = 0, 1, 2, 3$, we compute

$$f_Y(0) = \frac{1}{12} + \frac{1}{6} + \frac{1}{24} = \frac{7}{24}$$
$$f_Y(1) = \frac{1}{4} + \frac{1}{4} + \frac{1}{40} = \frac{21}{40}$$
$$f_Y(2) = \frac{1}{8} + \frac{1}{20} = \frac{7}{40}$$
$$f_Y(3) = \frac{1}{120}$$

Let us now solve parts (c) and (d).

(c) Let $f(x, y)$ be the joint probability distribution of X and Y (as given in the table in Exercise 42). To find the conditional distribution of X given $Y = 1$, we compute

$$f_{X|Y}(x|1) = \frac{f(x,1)}{f_Y(1)}, \qquad \text{for } x = 0, 1, 2.$$

(Once again, note that $f_Y(1)$ must be different than zero for the conditional distribution to make sense. In this case, it does make sense since $f_Y(1) = \frac{21}{40}$).
Carrying out the computations, we obtain

$$f_{X|Y}(0|1) = \frac{f(0,1)}{f_Y(1)} = \frac{\frac{1}{4}}{\frac{21}{40}} = \frac{10}{21}$$
$$f_{X|Y}(1|1) = \frac{f(1,1)}{f_Y(1)} = \frac{\frac{1}{4}}{\frac{21}{40}} = \frac{10}{21}$$
$$f_{X|Y}(2|1) = \frac{f(2,1)}{f_Y(1)} = \frac{\frac{1}{40}}{\frac{21}{40}} = \frac{1}{21}$$

(d) To find the conditional distribution of Y given $X = 0$, we compute

$$f_{Y|X}(y|0) = \frac{f(0,y)}{f_X(0)}, \qquad \text{for } y = 0, 1, 2, 3.$$

(Again, $f_X(0)$ must be different than zero for the conditional distribution to make sense. In this case, it does make sense since $f_X(0) = \frac{7}{15}$).
Carrying out the computations, we obtain

$$f_{Y|X}(0|0) = \frac{f(0,0)}{f_X(0)} = \frac{\frac{1}{12}}{\frac{7}{15}} = \frac{5}{28}$$
$$f_{Y|X}(1|0) = \frac{f(0,1)}{f_X(0)} = \frac{\frac{1}{4}}{\frac{7}{15}} = \frac{15}{28}$$
$$f_{Y|X}(2|0) = \frac{f(0,2)}{f_X(0)} = \frac{\frac{1}{8}}{\frac{7}{15}} = \frac{15}{56}$$
$$f_{Y|X}(3|0) = \frac{f(0,3)}{f_X(0)} = \frac{\frac{1}{120}}{\frac{7}{15}} = \frac{1}{56}.$$

4. Exercise 71 (d-e): Given the joint probability distribution

$$f(x, y, z) = \frac{xyz}{108}, \qquad \text{for } x = 1, 2, 3;\ y = 1, 2, 3;\ z = 1, 2$$

find

(d) the conditional distribution of Z given $X = 1$ and $Y = 2$;

(e) the joint conditional distribution of Y and Z given $X = 3$.

Solution: In Recitation 5, we solved parts (a), (b), (c) as follows.

(a) Let $f_{X,Y}(x, y)$ be the joint marginal distribution of X and Y. Then,

$$f_{X,Y}(x, y) = \sum_{z=1}^{2} f(x, y, z) = \sum_{z=1}^{2} \frac{xyz}{108} = \frac{xy}{36}, \qquad \text{for } x = 1, 2, 3;\ y = 1, 2, 3.$$

(b) Let $f_{X,Z}(x, z)$ be the joint marginal distribution of X and Z. Then, we compute it as

$$f_{X,Z}(x, z) = \sum_{y=1}^{3} f(x, y, z) = \sum_{y=1}^{3} \frac{xyz}{108} = \frac{xz}{18}, \qquad \text{for } x = 1, 2, 3;\ z = 1, 2.$$

(c) Let $f_X(x)$ the marginal distribution of X. We can compute it as

$$f_X(x) = \sum_{y=1}^{3}\sum_{z=1}^{2} f(x, y, z) = \sum_{y=1}^{3}\sum_{z=1}^{2} \frac{xyz}{108} = \sum_{y=1}^{3} \frac{xy}{36} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3,$$

Equivalently,

$$f_X(x) = \sum_{y=1}^{3} g(x, y) = \sum_{y=1}^{3} \frac{xy}{36} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3.$$

or

$$f_X(x) = \sum_{z=1}^{2} h(x, z) = \sum_{z=1}^{2} \frac{xz}{18} = \frac{x}{6}, \qquad \text{for } x = 1, 2, 3.$$

Let us now solve parts (d) and (e).

(d) To find the conditional distribution of Z given $X = 1$ and $Y = 2$, we compute

$$f_{Z|X,Y}(z|1, 2) = \frac{f(1, 2, z)}{f_{X,Y}(1, 2)}, \qquad \text{for } z = 1, 2.$$

($f_{X,Y}(1, 2)$ must be different than zero for the conditional distribution to make sense. It is easy to check that $f_{X,Y}(1, 2) = 1/18 \neq 0$.)
Using the explicit form of $f(x, y, z)$, we obtain

$$f_{Z|X,Y}(z|1, 2) = \frac{\frac{2z}{108}}{\frac{1}{18}} = \frac{z}{3}, \qquad \text{for } z = 1, 2.$$

(e) To find the joint conditional distribution of Y and Z given $X = 3$, we compute

$$f_{Y,Z|X}(y, z|3) = \frac{f(3, y, z)}{f_X(3)}, \qquad \text{for } y = 1, 2, 3;\ z = 1, 2.$$

($f_X(3)$ must be different than zero for the conditional distribution to make sense. It is easy to check that $f_X(3) = 1/2 \neq 0$.)

Using the explicit form of $f(x, y, z)$, we obtain

$$f_{Y,Z|X}(y, z|3) = \frac{\frac{3yz}{108}}{\frac{1}{2}} = \frac{yz}{18}, \qquad \text{for } y = 1, 2, 3;\ z = 1, 2.$$

5. Exercises 74 b & 75 b: If the joint density of X and Y is given by

$$f(x, y) = \begin{cases} \frac{1}{4}(2x + y) & \text{for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

find

74 b) the conditional density of Y given $X = 1/4$;

75 b) the conditional density of X given $Y = 1$.

Solution:

Note for TA's: Remind the students that we can rewrite the joint density of X and Y in terms of the indicator function

$$1_{(0,1)\times(0,2)}(x, y) = \begin{cases} 1 & \text{for } 0 < x < 1,\ 0 < y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

as

$$f(x, y) = \frac{1}{4}(2x + y)\ 1_{(0,1)\times(0,2)}(x, y) \qquad \text{for } -\infty < x < \infty,\ -\infty < y < \infty.$$

This is in line with the definition of $1_A(x)$ given earlier; we simply consider A as a subset of $\mathbb{R}^2$ and x as a point in $\mathbb{R}^2$. In this question, our set is $(0, 1) \times (0, 2)$. Note that, in this case, we can write the indicator function as the product of $1_{(0,1)}(x)$ and $1_{(0,2)}(y)$. However, this decomposition (as a product) may not hold for all subset of $\mathbb{R}^2$. Consider the set $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1, x + y < 1\}$ given in question 2 above. We can not write

$$1_S(x, y) = \begin{cases} 1 & \text{for } (x, y) \in S \\ 0 & \text{elsewhere} \end{cases}$$

as the product of $1_B(x)$ and $1_C(y)$ for some subsets B and C of $\mathbb{R}$. (end of the note)

In Recitation 5, we solved part 74 a & 75 a as follows:

74 a) Let $f_X(x)$ be the marginal density of X. Clearly, for $x \notin (0, 1)$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = 0.$$

For $x \in (0,1)$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^2 \frac{1}{4}(2x+y)\, dy = \frac{1}{4}(4x+2) = \frac{1}{2}(2x+1).$$

Hence,

$$f_X(x) = \frac{1}{2}(2x+1)\ 1_{(0,1)}(x), \qquad \text{for } -\infty < x < \infty$$

in terms of the indicator function

$$1_{(0,1)}(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{for } x \notin (0,1) \end{cases}$$

Clearly we can also write the marginal density of X as

$$f_X(x) = \begin{cases} \frac{1}{2}(2x+1) & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

75 a) Let $f_Y(y)$ be the marginal density of Y. Likewise, for $y \notin (0,2)$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = 0.$$

For $y \in (0,2)$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{1}{4}(2x+y)\, dx = \frac{1}{4}(1+y).$$

Hence,

$$f_Y(y) = \frac{1}{4}(1+y)\ 1_{(0,2)}(y) \qquad \text{for } -\infty < y < \infty.$$

Now let us solve 74 b & 75 b.

74 b) The function

$$f_{Y|X}\left(y \,\middle|\, \frac{1}{4}\right) = \frac{f\left(\frac{1}{4}, y\right)}{f_X\left(\frac{1}{4}\right)}, \qquad \text{for } -\infty < y < \infty$$

gives the conditional density of Y given $X = 1/4$ (again note that $f_X(1/4) = 3/4 \neq 0$).
For $y \notin (0,2)$, $f(\frac{1}{4}, y) = 0$, and therefore $f_{Y|X}(y\,|\,\frac{1}{4}) = 0$.
For $y \in (0,2)$, we have $f(\frac{1}{4}, y) = \frac{1}{4}(\frac{1}{2} + y)$, and this gives

$$f_{Y|X}\left(y \,\middle|\, \frac{1}{4}\right) = \frac{\frac{1}{4}(\frac{1}{2}+y)}{\frac{3}{4}} = \frac{1}{3}\left(\frac{1}{2}+y\right) = \frac{1}{6}(1+2y).$$

Hence, we have

$$f_{Y|X}\left(y \,\middle|\, \frac{1}{4}\right) = \frac{1}{6}(1+2y)\ 1_{(0,2)}(y), \qquad \text{for } -\infty < x < \infty.$$

75 b) The function

$$f_{X|Y}(x\,|1) = \frac{f(x,1)}{f_Y(1)}, \qquad \text{for } -\infty < x < \infty$$

gives the conditional density of X given $Y = 1$ (again note that $f_Y(1) = 1/2 \neq 0$).
For $x \notin (0,1)$, $f(x,1) = 0$, and therefore $f_{X|Y}(x\,|1) = 0$.
For $x \in (0,1)$, we have $f(x,1) = \frac{1}{4}(2x+1)$, and this gives

$$f_{X|Y}(x\,|1) = \frac{\frac{1}{4}(2x+1)}{\frac{1}{2}} = \frac{1}{2}(2x+1).$$

Hence, we have

$$f_{X|Y}(x\,|1) = \frac{1}{2}(2x+1)\; 1_{(0,1)}(x), \qquad \text{for } -\infty < x < \infty.$$

6. Let X and Y be two continuous random variables with the joint density

$$f(x,y) = \begin{cases} \dfrac{1}{y} & \text{for } 0 < x < y,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

(a) the conditional density of Y given $X = 1/2$.

(b) the conditional density of X given $Y = 1/4$.

Solution: The marginal densities of X and Y are computed in Recitation 5 (see the solution of Exercise 77). We obtained these functions last recitation as follows:

74 a) Let $f_X(x)$ be the marginal density of X. Clearly, for $x \notin (0,1)$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = 0.$$

For $x \in (0,1)$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 \frac{1}{y}\,dy = \ln y\Big|_x^1 = -\ln x.$$

Hence, we have

$$f_X(x) = -\ln(x)\; 1_{(0,1)}(x), \qquad \text{for } -\infty < x < \infty.$$

75 a) Let $f_Y(y)$ be the marginal density of Y. Likewise, for $y \notin (0,1)$,

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = 0.$$

For $y \in (0,1)$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y \frac{1}{y}\,dx = 1.$$

Hence,

$$f_Y(y) = 1_{(0,1)}(x), \qquad \text{for } -\infty < x < \infty.$$

Now let us solve our exercise.

(a) The function

$$f_{Y|X}\left(y\,\Big|\,\frac{1}{2}\right) = \frac{f\left(\frac{1}{2}, y\right)}{f_X\left(\frac{1}{2}\right)}, \qquad \text{for } -\infty < y < \infty$$

gives the conditional density of Y given $X = 1/2$ (again note that $f_X(1/2) = \ln 2 \neq 0$), Note for TA's: remind the students that $\ln 2 > 0$.

For $y \notin (\frac{1}{2}, 1)$, $f(\frac{1}{2}, y) = 0$, and therefore $f_{Y|X}(y\,|\,\frac{1}{2}) = 0$.
For $y \in (\frac{1}{2}, 1)$, we have $f(\frac{1}{2}, y) = \frac{1}{y}$, and this gives

$$f_{Y|X}\left(y\,\Big|\,\frac{1}{2}\right) = \frac{\frac{1}{y}}{\ln 2} = \frac{1}{y \ln 2}.$$

Hence, we have

$$f_{Y|X}\left(y\,\Big|\,\frac{1}{2}\right) = \frac{1}{y \ln 2}\, 1_{(\frac{1}{2},1)}(y), \qquad \text{for } -\infty < y < \infty.$$

(b) The function

$$f_{X|Y}\left(x\,\Big|\,\frac{1}{4}\right) = \frac{f(x, \frac{1}{4})}{f_Y(\frac{1}{4})}, \qquad \text{for } -\infty < x < \infty$$

gives the conditional density of X given $Y = 1/4$ (again note that $f_Y(1/4) = 1 \neq 0$).
For $x \notin (0, \frac{1}{4})$, $f(x, \frac{1}{4}) = 0$, and therefore $f_{X|Y}(x\,|\,\frac{1}{4}) = 0$.
For $x \in (0, \frac{1}{4})$, we have $f(x, \frac{1}{4}) = 4$, and this gives

$$f_{X|Y}\left(x\,\Big|\,\frac{1}{4}\right) = \frac{4}{1} = 4.$$

Hence, we have

$$f_{X|Y}\left(x\,\Big|\,\frac{1}{4}\right) = 4 \cdot 1_{(0,\frac{1}{4})}(x), \qquad \text{for } -\infty < x < \infty.$$

7. Exercise 78: With reference to Example 22 (see page 94), find

(a) the conditional density of X_2 given $X_1 = \frac{1}{3}$ and $X_3 = 2$;

(b) the joint conditional density of X_2 and X_3 given $X_1 = \frac{1}{2}$.

Solution:

(a) In Example 22, the joint marginal density of X_1 and X_3 is computed as

$$f_{X_1,X_3}(x_1, x_3) = \begin{cases} \left(x_1 + \frac{1}{2}\right)e^{-x_3} & \text{for } 0 < x_1 < 1,\ x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Note for TA's: If you are not convinced that the students have a good understanding of marginals (if you think you have enough time), you may go over the derivation of the joint marginal density of X_1 and X_3 given in Example 22.

The conditional density of X_2 given $X_1 = \frac{1}{3}$ and $X_3 = 2$ is given by the function

$$f_{X_2|X_1,X_3}\left(x_2 \,\Big|\, \frac{1}{3}, 2\right) = \frac{f(\frac{1}{3}, x_2, 2)}{f_{X_1,X_3}(\frac{1}{3}, 2)}, \qquad \text{for } -\infty < x_2 < \infty.$$

(Again, note that $f_{X_1,X_3}(\frac{1}{3}, 2) = \frac{5}{6}e^{-2} \neq 0$.)
For $x_2 \notin (0,1)$, $f(\frac{1}{3}, x_2, 2) = 0$, and therefore $f_{X_2|X_1,X_3}(x_2 \,|\, \frac{1}{3}, 2) = 0$.
For $x_2 \in (0,1)$, we have $f(\frac{1}{3}, x_2, 2) = (\frac{1}{3} + x_2)e^{-2}$, and this gives

$$f_{X_2|X_1,X_3}\left(x_2 \,\Big|\, \frac{1}{3}, 2\right) = \frac{(\frac{1}{3} + x_2)e^{-2}}{\frac{5}{6}e^{-2}} = \frac{6}{5}\left(\frac{1}{3} + x_2\right) = \frac{2}{5}\left(1 + 3x_2\right).$$

Hence, we have

$$f_{X_2|X_1,X_3}\left(x_2 \,\Big|\, \frac{1}{3}, 2\right) = \begin{cases} \frac{2}{5}\left(1 + 3x_2\right) & \text{for } 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(b) In Example 22, the marginal density of X_1 is computed as

$$g(x_1) = \begin{cases} x_1 + \frac{1}{2} & \text{for } 0 < x_1 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Note for TA's: If you are not convinced that the students have a good understanding of marginals (if you think you have enough time), you may go over the derivation of the joint marginal density of X_1 given in Example 22.

The joint conditional density of X_2 and X_3 given $X_1 = \frac{1}{2}$ is given by the function

$$f_{X_2,X_3|X_1}\left(x_2, x_3 \,\Big|\, \frac{1}{2}\right) = \frac{f(\frac{1}{2}, x_2, x_3)}{f_{X_1}(\frac{1}{2})}, \qquad \text{for } -\infty < x_2 < \infty \text{ and } -\infty < x_3 < \infty.$$

(Again, note that $f_{X_1}(\frac{1}{2}) = 1 \neq 0$.)
For $x_2 \notin (0,1)$ or $x_3 \leq 0$, $f(\frac{1}{2}, x_2, x_3) = 0$, and therefore $f_{X_2,X_3|X_1}(x_2, x_3 \,|\, \frac{1}{2}) = 0$.
For $x_2 \in (0,1)$ and $x_3 > 0$, we have $f(\frac{1}{2}, x_2, x_3) = (\frac{1}{2} + x_2)e^{-x_3}$, and this gives

$$f_{X_2,X_3|X_1}\left(x_2, x_3 \,\Big|\, \frac{1}{2}\right) = \frac{(\frac{1}{2} + x_2)e^{-x_3}}{1} = \left(\frac{1}{2} + x_2\right)e^{-x_3}.$$

Hence, we have

$$f_{X_2,X_3|X_1}\left(x_2, x_3 \,\Big|\, \frac{1}{2}\right) = \begin{cases} \left(\frac{1}{2} + x_2\right)e^{-x_3} & \text{for } 0 < x_2 < 1 \text{ and } x_3 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

8. Let X and Y be two continuous random variables whose joint distribution function is given by

$$F(x, y) = \begin{cases} (1 - e^{-x^2})(1 - e^{-y^2}) & \text{for } x > 0,\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Determine whether X and Y are independent.

Solution: Let $F_X(x)$ be the distribution function of X, and let $F_Y(y)$ be the distribution function of Y. Note that we have $F_X(x) = F(x, \infty)$ for all $-\infty < x < \infty$, and $F_Y(y) = F(\infty, y)$ for all $-\infty < y < \infty$. That is, we have

$$F_X(x) = \begin{cases} (1 - e^{-x^2}) & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_Y(y) = \begin{cases} (1 - e^{-y^2}) & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

It is easy to verify that we have $F(x, y) = F_X(x)F_Y(y)$, for all $-\infty < x < \infty$ and $-\infty < y < \infty$. Hence, X and Y are independent.

9. With reference to Exercise 42, determine whether X and Y are independent.

   Solution: We already computed the marginal distribution $f_X(x)$ of X and the marginal distribution $f_Y(y)$ of Y. (see the solution of Exercise 70 above).

   For example with $x = 1$ and $y = 3$, we have

$$f_X(1) = \frac{7}{15} \quad \text{and} \quad f_Y(3) = \frac{1}{120}$$

   whereas $f(1, 3) = 0$. Hence X and Y are not independent.

10. Let X and Y be two independent random variables with the marginal densities

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(y) = \begin{cases} e^{-y} & \text{for } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   Find

   (a) the distribution function of $Z = X + Y$;

   (b) the density of Z.

   Solution: Note that since X and Y are independent, their joint density is given by the product of marginal densities. That is, if we let $f(x, y)$ be their joint density, then we have $f(x, y) = f_X(x)f_Y(y)$ for all $-\infty < x < \infty$ and $-\infty < y < \infty$. Using the marginal densities given in the question, we can write the function $f(x, y)$ as

$$f(x, y) = \begin{cases} e^{-(x+y)} & \text{for } x > 0 \text{ and } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Let $F_Z(z)$ be the distribution of Z. Clearly Z takes values on $(0, \infty)$. Hence, for $z \leq 0$, $F_Z(z) = 0$.

For $z > 0$, we compute

$$\begin{aligned} F_Z(z) = \mathbb{P}(Z \le z) &= \mathbb{P}(X + Y \le z) \\ &= \int_0^z \int_0^{z-x} f_{X,Y}(x,y)dy\,dx \\ &= \int_0^z \int_0^{z-x} e^{-(x+y)}dy\,dx \\ &= \int_0^z e^{-x} \int_0^{z-x} e^{-y}dy\,dx \\ &= \int_0^z e^{-x}(1 - e^{-(z-x)})\,dx \\ &= \int_0^z (e^{-x} - e^{-z})\,dx = 1 - e^{-z} - ze^{-z}. \end{aligned}$$

Hence we have

$$F_Z(z) = \begin{cases} 1 - e^{-z} - ze^{-z} & \text{for } z > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Note that the distribution function for $z > 0$ can also be obtained using the convolution formula discussed in class. That is;

$$F_Z(z) = \int_0^z f_X(x)F_Y(z-x)dx,$$

where $F_Y(y)$ denotes the distribution function of Y. For $y \le 0$, $F_Y(y) = \int_{-\infty}^y f_Y(u)\,du$ is obviously zero, and for $y > 0$ we have

$$F_Y(y) = \int_{-\infty}^z f_Y(u)du = \int_0^z e^{-u}du = 1 - e^{-z}.$$

Hence we have $F_Y(y) = (1 - e^{-z})1_{(0,\infty)}(y)$.

When we go back to the convolution formula (for the distribution function), we obtain

$$F_Z(z) = \int_0^z f_X(x)F_Y(z-x)dx = \int_0^z e^{-x}(1 - e^{-(z-x)})dx$$

and this gives the same result for $z > 0$.

(b) Let $f_Z(z)$ the density of Z. Note that we have $f_Z(z) = F_Z'(z)$ wherever $F_Z(z)$ is differentiable.

$$\begin{aligned} &\text{For } z > 0, \quad F_Z'(z) = e^{-z} - e^{-z} + ze^{-z} = ze^{-z} \\ &\text{For } z < 0, \quad F_Z'(x) = 0. \end{aligned}$$

The assignment at the point $z = 0$ does not matter. We can set it to zero for convenience. Hence, we write

$$f_Z(z) = \begin{cases} ze^{-z} & \text{for } z > 0 \\ 0 & \text{for } z \le 0 \end{cases}$$

The density function of Z can also be obtained directly without finding the distribution first. For $z > 0$, the density can be found using the convolution formula (for the density function) as

$$f_Z(z) = \int_0^z f_X(x)f_Y(z-x)dx = \int_0^z e^{-x}e^{-(z-x)}dx = \int_0^z e^{-z}\,dx = ze^{-z}.$$

Since both X and Y take values on $(0, \infty)$, the density function $f_Z(z)$ for $z \le 0$ can be immediately written as zero.