Recitation IV

Continuous random variables (r.v.'s):

* The probabilities related to a continuous r.v. are determined through its probability density function (p.d.f.) $f(x)$ satisfying

i/ $f(x) \geq 0$ ii) $\int_{-\infty}^{+\infty} f(x)dx = 1$

Then $\mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X \leq b) = \mathbb{P}(a \leq X < b)$

$= \mathbb{P}(a < X < b) = \int_a^b f(x)\,dx.$

Also, $\mathbb{P}(X = x) = 0$ for any real number x.

Ex 3.22 pg 90: The p.d.f. of the random variable X is given by

$$f(x) = \begin{cases} \frac{c}{\sqrt{x}} & \text{for } 0 < x < 4 \\ 0 & \text{elsewhere} \end{cases}$$

Find

a) the value of c

b) $\mathbb{P}(X > 1)$.

Sol: $$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{0} \underbrace{f(x)}_{0}dx + \int_0^4 f(x)\,dx + \int_4^{\infty} \underbrace{f(x)}_{0}dx = \int_0^4 \frac{c}{\sqrt{x}}\,dx$$

$$= c\,(2\sqrt{x})\Big|_{x=0}^{x=4} = 4c$$

So we set $c = \frac{1}{4}$

b) $\mathbb{P}(X>1) = \mathbb{P}(1<X<4) = \int_1^4 f(x)dx = \int_1^4 \frac{1}{4}\frac{1}{\sqrt{x}}dx$

$$= \frac{1}{4}\left(2\sqrt{x}\right)\Big|_{x=1}^{x=4} = \frac{1}{4}(4-2) = \frac{1}{2}$$

Ex 3.24 pg 90 : The p.d.f of the random variable Z is given by

$$f(z) = \begin{cases} kze^{-z^2} & \text{for } z>0 \\ 0 & \text{for } z\le 0 \end{cases}$$

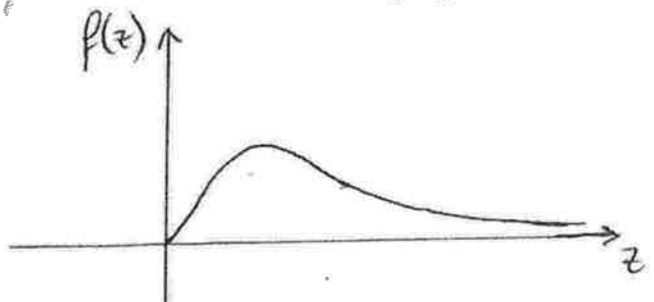
Find k.

Sol: Note that $\frac{d}{dz}\left(e^{-z^2}\right) = -2ze^{-z^2}$. Hence

$$\int_{-\infty}^{+\infty} f(z)dz = \int_0^{\infty} kze^{-z^2}dz = \left(-\frac{k}{2}\right)\int_0^{\infty}(-2z)e^{-z^2}dz$$

$$= -\frac{k}{2}\left(e^{-z^2}\right)\Big|_{z=0}^{z=\infty} = -\frac{k}{2}(0-1) = \frac{k}{2}$$

Hence, $k=2$. The graph looks like

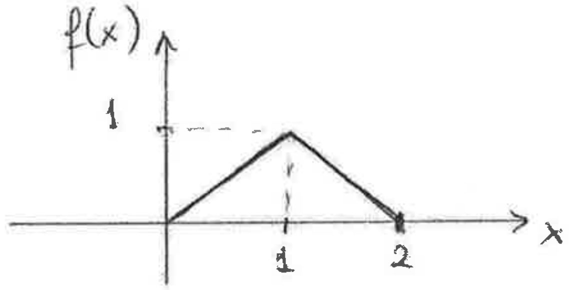


Ex 3.29 pg 91 : Find the cumulative distribution function (cdf) of the random variable X whose probability density is given by

$$f(x) = \begin{cases} x & \text{for } 0<x<1 \\ 2-x & \text{for } 1\le x<2 \\ 0 & \text{elsewhere} \end{cases}$$

Also sketch the graphs of the probability density (pdf) and cumulative distribution functions (cdf).

Sol:



Let's now find the c.d.f:

For $x<0$, $F(x) = \int_{-\infty}^{x} f(z)dz = 0$.

For $0\leq x<1$, $F(x) = \int_{-\infty}^{x} f(z)dz = \int_{0}^{x} z\,dz = \left.\frac{z^2}{2}\right|_{z=0}^{z=x} = \frac{x^2}{2}$

For $1\leq x<2$, $F(x) = \int_{-\infty}^{x} f(z)dz = \int_{0}^{1} z\,dz + \int_{1}^{x}(2-z)dz$

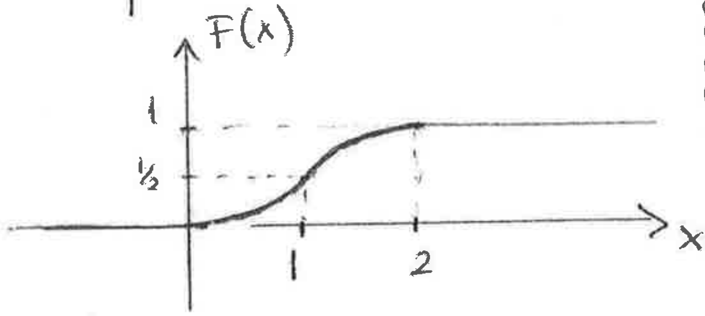
$$= \frac{1}{2} + \left(2z - \frac{z^2}{2}\right)_{z=1}^{z=x} = \frac{1}{2} + \left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= -\frac{x^2}{2} + 2x - 1$$

For $2 \leq x$, $F(x) = \int_{-\infty}^{x} f(z)dz = \int_{0}^{2} f(z)dz = 1$,

which is the area under the curve.

*/ When we plot $F(x)$, we get



So, we have

$$F(x) = \begin{cases} 0 & \text{for } x<0 \\ \frac{x^2}{2} & \text{for } 0 \leq x < 1 \\ -\frac{x^2}{2} + 2x - 1 & \text{for } -1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}$$

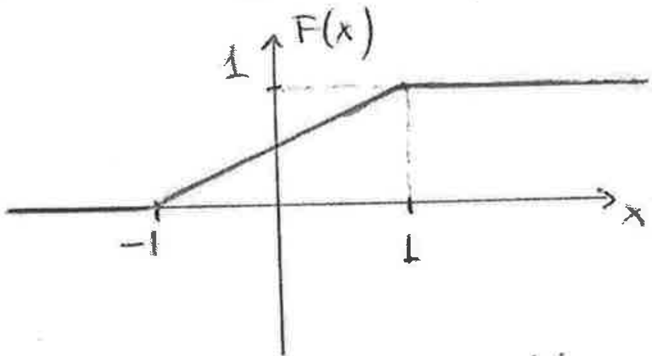
Ex 3.32 pg 91: The c.d.f. of the random variable X is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

a) Find $f(x)$.

b) Compute $\mathbb{P}(-\frac{1}{2} < X < \frac{1}{2})$ and $\mathbb{P}(2 < X < 3)$.

Sol:



a) It is a continuous random variable, hence the pdf can be computed as $f(x) = F'(x)$ at the points where

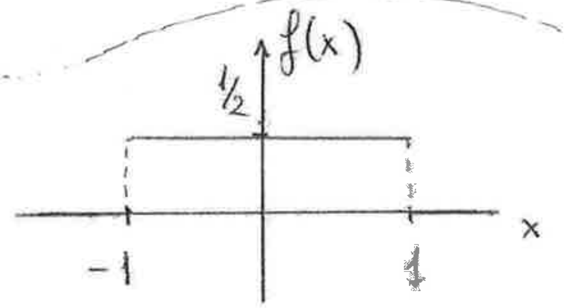
$F(x)$ is differentiable.

For $x < -1$, $f(x) = F'(x) = 0$

For $-1 < x < 1$, $f(x) = \frac{1}{2}$

For $1 < x$, $f(x) = 0$.

The assignments at the points $x = -1$ and $x = 1$ does not matter. We can set them to zero for convenience. Hence we get



$$f(x) = \begin{cases} \frac{1}{2} & \text{for } -1<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

b) $$\mathbb{P}\left(-\frac{1}{2} < X < \frac{1}{2}\right) = \int_{-\frac{1}{2}}^{1/2} f(x)\,dx = \int_{-\frac{1}{2}}^{1/2} \frac{1}{2}\,dx = \frac{1}{2}$$

$$\mathbb{P}(2 < X < 3) = \int_2^3 f(x)\,dx = \int_2^3 0\,dx = 0.$$

Ex 3.37 pg 92: The c.d.f. of a continuous random variable X is given by

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find $\mathbb{P}(X \leq 2)$, $\mathbb{P}(1 < X < 3)$ and $\mathbb{P}(X > 4)$.

Sol:

*/ $\mathbb{P}(X \leq 2) = F(2) = 1 - 3e^{-2}$

*/ Since X is a continuous random variable we have $\mathbb{P}(1 < X < 3) = \mathbb{P}(1 < X \leq 3)$. So, we compute

$$\mathbb{P}(1 < X < 3) = \mathbb{P}(1 < X \leq 3) = \mathbb{P}(X \leq 3) - \mathbb{P}(X \leq 1)$$
$$= F(3) - F(1) = (1 - 4e^{-3}) - (1 - 2e^{-1})$$
$$= 2e^{-1} - 4e^{-3}$$

*/ $\mathbb{P}(X > 4) = 1 - \mathbb{P}(X \leq 4) = 1 - F(4)$
$$= 1 - (1 - 5e^{-4}) = 5e^{-4}.$$

Ch 4

Exercise: Let X be a continuous random variable whose density is given by

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 \leq x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

a/ Compute $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$

b/ Use part a/ to compute $\mathbb{E}\left[(X+1)^2\right]$

Sol: a/ We have

$$\mathbb{E} X = \int_{-\infty}^{+\infty} x f(x) dx \quad \text{and} \quad \mathbb{E}(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx.$$

Let's compute the first one:

$$\mathbb{E} X = \int_{-\infty}^{+\infty} x f(x) dx = \int_0^1 x \cdot x \, dx + \int_1^2 x \cdot (2-x) dx$$

[redacted]

$$= \left(\frac{x^3}{3}\right)\Big|_{x=0}^{x=1} + \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{3} + \left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right) = 1.$$

Next, we compute

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \left(\frac{x^4}{4}\right)\Big|_{x=0}^{x=1} + \left(\frac{2}{3}x^3 - \frac{x^4}{4}\right)\Big|_{x=1}^{x=2}$$

$$= \frac{1}{4} + \left(\frac{2}{3}\cdot 8 - 4\right) - \left(\frac{2}{3} - \frac{1}{4}\right) = \frac{7}{6}$$

b/ Note that $(X+1)^2 = X^2 + 2X + 1$. So

$$\mathbb{E}[(X+1)^2] = \mathbb{E}[X^2 + 2X + 1] = \mathbb{E}[X^2] + 2\,\mathbb{E}[X] + 1.$$

Using part a, we get

$$\mathbb{E}[(X+1)^2] = \frac{7}{6} + 2\cdot 1 + 1 = \frac{25}{6}$$

Exercise: Let X be a discrete random variable with the probability distribution

$$f(x) = \frac{|x-2|}{7} \quad \text{for } x \in \{-1, 0, 1, 3\}$$

a/ Compute $\mathbb{E}X$

b/ Compute $\mathbb{E}\, e^{tX}$ as a function of t.

Solution: a/ We apply the definition of the expected value to compute

$$\mathbb{E}X = \sum_x x f(x) = (-1)\cdot f(-1) + 0 \cdot f(0) + 1 \cdot f(1) + 3 \cdot f(3)$$

$$= -1 \cdot \frac{3}{7} + 0 + 1 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7} = \frac{1}{7}$$

b) We compute

$$\mathbb{E}\left(e^{tX}\right) = \sum_x e^{tx} f(x)$$

$$= e^{-t} f(-1) + e^{0.t} f(0) + e^{t} f(1) + e^{3t} f(3)$$

$$= \frac{3}{7} e^{-t} + \frac{2}{7} + \frac{1}{7} e^{t} + \frac{1}{7} e^{3t}$$

$$= \frac{1}{7}\left(2 + 3e^{-t} + e^{t} + e^{3t}\right)$$